

Sequences Defined Recursively

Definition: The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there exists an integer N such that for all n ,

$$\text{if } n > N \text{ then } |a_n - L| < \varepsilon.$$

If no such number L exists, we say $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L , we write $\lim_{n \rightarrow \infty} a_n = L$ or $a_n \rightarrow L$ and call L the **limit** of the sequence.

Some sequences are defined **recursively**. Recursive definitions enable us to compute the first, second, third, ... n th term, but we cannot compute the n th term without first computing the first $n - 1$.

The **Fibonacci sequence** is a perfect example for this.

Definition: The **Fibonacci sequence** is defined recursively as

$$F_1 = 1, \quad F_2 = 1, \quad \text{and for all } n \in \mathbb{N}, \quad F_{n+2} = F_n + F_{n+1}$$

The first few terms of the Fibonacci Sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... The explicit formula for the n th term of this sequence is a very interesting formula.

The following method can be used to find the limit of a recursive sequence - provided that we have already established that the limit exists.

Example 1. Consider the sequence defined recursively: $a_1 = 2$ and $a_{n+1} = \frac{1}{2}a_n + 3$. Assume that this limit exists and find its value.

Solution: This method is based on the fact that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1}$. Let us denote this limit by x .

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} a_{n+1} \\ \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \left(\frac{1}{2}a_n + 3 \right) \end{aligned}$$

Using properties of limits, we re-write the right-hand side:

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2}a_n + 3 \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{2}a_n \right) + \lim_{n \rightarrow \infty} 3 = \frac{1}{2} \lim_{n \rightarrow \infty} a_n + 3$$

So now we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \frac{1}{2} \lim_{n \rightarrow \infty} a_n + 3 \\ x &= \frac{1}{2}x + 3 && \text{subtract } \frac{1}{2}x \\ \frac{1}{2}x &= 3 && \text{multiply by 2} \\ x &= 6 \end{aligned}$$

So this limit is 6.

Example 2. Consider the sequence defined recursively: $a_1 = 0$, $a_2 = 1$, and $a_{n+1} = a_n + 2a_{n-1}$. Assume that the limit $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ exists and find its value.

Solution: This method is based on the fact that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{a_n}{a_{n-1}}$. Let us denote this limit by x .

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{a_n + 2a_{n-1}}{a_n} = \lim_{n \rightarrow \infty} \frac{a_n}{a_n} + \lim_{n \rightarrow \infty} \frac{2a_{n-1}}{a_n} = 1 + 2 \lim_{n \rightarrow \infty} \frac{a_{n-1}}{a_n}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1 + 2 \lim_{n \rightarrow \infty} \frac{a_{n-1}}{a_n}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1 + \frac{2}{\lim_{n \rightarrow \infty} \frac{a_n}{a_{n-1}}}$$

$$x = 1 + \frac{2}{x}$$

multiply by x

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0 \implies x_1 = 2 \quad x_2 = -1$$

since all terms of this sequence are positive, the limit is the positive solution, $x = 2$.

Example 3. Consider the sequence $a_1 = 1$ and $a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right)$. Assume that the limit $\lim_{n \rightarrow \infty} a_n$ exists and find its value.

Solution: This method is based on the fact that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1}$. Let us denote this limit by x .

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{1}{2} \left(a_n + \frac{2}{a_n} \right) \right)$$

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{2} \lim_{n \rightarrow \infty} \left(a_n + \frac{2}{a_n} \right)$$

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{2} \left(\lim_{n \rightarrow \infty} a_n + \frac{2}{\lim_{n \rightarrow \infty} a_n} \right)$$

$$x = \frac{1}{2} \left(x + \frac{2}{x} \right)$$

$$x = \frac{1}{2}x + \frac{1}{x}$$

subtract $\frac{1}{2}x$

$$\frac{1}{2}x = \frac{1}{x}$$

multiply by x

$$\frac{1}{2}x^2 = 1$$

multiply by 2

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

Since all terms of the sequence are positive, the negative solution is ruled out and the limit must be $\sqrt{2}$.

Example 4. Consider the sequence $a_1 = 5$ and $a_{n+1} = \frac{a_n + 3}{a_n + 1}$. Assume that the limit $\lim_{n \rightarrow \infty} a_n$ exists and find its value.

Solution: This method is based on the fact that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1}$. Let us denote this limit by x .

$$\begin{aligned}
 \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} a_{n+1} \\
 \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{a_n + 3}{a_n + 1} \\
 \lim_{n \rightarrow \infty} a_n &= \frac{\lim_{n \rightarrow \infty} a_n + 3}{\lim_{n \rightarrow \infty} a_n + 1} \\
 x &= \frac{x + 3}{x + 1} && \text{multiply by } x + 1 \\
 x(x + 1) &= x + 3 \\
 x^2 + x &= x + 3 \\
 x^2 &= 3 \\
 x &= \pm\sqrt{3}
 \end{aligned}$$

Since all terms of the sequence are positive, the negative solution is ruled out and the limit must be $\sqrt{3}$.

Example 5. Compute the value of $\sqrt{20 + \sqrt{20 + \sqrt{20 + \dots}}}$

Solution: Define $a_1 = 20$ and $a_{n+1} = \sqrt{20 + a_n}$. Assuming the limit exists, $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1}$. Let us denote this limit by x .

$$\begin{aligned}
 \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} a_{n+1} \\
 \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \sqrt{20 + a_n}
 \end{aligned}$$

We will use a theorem here we did not yet prove: that $\lim_{n \rightarrow \infty} \sqrt{c_n} = \sqrt{\lim_{n \rightarrow \infty} c_n}$. Let us assume this property for now. Using properties of limits,

$$\lim_{n \rightarrow \infty} \sqrt{20 + a_n} = \sqrt{\lim_{n \rightarrow \infty} (20 + a_n)} = \sqrt{\lim_{n \rightarrow \infty} 20 + \lim_{n \rightarrow \infty} a_n} = \sqrt{20 + \lim_{n \rightarrow \infty} a_n}$$

So now we have that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} a_n &= \sqrt{20 + \lim_{n \rightarrow \infty} a_n} \\
 x &= \sqrt{20 + x} \\
 x^2 &= x + 20 \\
 x^2 - x - 20 &= 0 \\
 (x - 5)(x + 4) &= 0 \quad \implies \quad x_1 = 5 \quad x_2 = -4
 \end{aligned}$$

Since all terms of the sequence are positive, the negative solution is ruled out and the limit must be 5.

Example 6. Let $F_1, F_2, F_3, \dots$ be the Fibonacci sequence defined by $F_1 = 1$, $F_2 = 1$, and for $F_{n+2} = F_n + F_{n+1}$ for $n \geq 1$. Compute $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n}$.

Solution: Let us denote $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n}$ by x . We will state that $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \lim_{n \rightarrow \infty} \frac{F_{n+2}}{F_{n+1}}$ and solve for x .

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} &= \lim_{n \rightarrow \infty} \frac{F_{n+2}}{F_{n+1}} \\ \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} &= \lim_{n \rightarrow \infty} \frac{F_n + F_{n+1}}{F_{n+1}} \\ \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} &= \lim_{n \rightarrow \infty} \left(\frac{F_n}{F_{n+1}} + \frac{F_{n+1}}{F_{n+1}} \right) \\ \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} &= \lim_{n \rightarrow \infty} \frac{F_n}{F_{n+1}} + 1 \\ x &= \frac{1}{x} + 1 \\ x^2 &= 1 + x \\ x^2 - x - 1 &= 0 \end{aligned}$$

$$x_{1,2} = \frac{1 \pm \sqrt{1 - (-4)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since all terms of the sequence are positive, the negative solution, $\frac{1 - \sqrt{5}}{2}$ is ruled out and so the limit must be $\frac{1 + \sqrt{5}}{2}$.

Practice Problems

Assume that the following sequences all have limits. In case of each of the sequences given, find the value of its limit.

1. $a_1 = 2$, $a_{n+1} = \frac{12}{1 + a_n}$

5. $a_1 = 3$, $a_{n+1} = 6 - \sqrt{a_n}$

2. $a_1 = -1$, $a_{n+1} = \frac{a_n + 25}{a_n + 1}$

6. $a_1 = 10$, $a_{n+1} = 4 + \frac{21}{a_n}$

3. $a_1 = -1$, $a_{n+1} = \sqrt{5 + 2a_n}$

7. $a_1 = 2$, $a_{n+1} = \frac{1}{2 + a_n}$

4. $a_1 = 2$, $a_{n+1} = \sqrt{2a_n}$

8. $a_1 = 1$, $a_{n+1} = \sqrt{1 + a_n}$

9. Consider the sequence defined recursively as $a_1 = 1$, $a_2 = 1$, $a_{n+2} = 2a_n + a_{n+1}$. Compute $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$.

10. Compute the value of the infinite continued fraction
$$2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

11. A farmer plants A acres of wheat one year. Each year thereafter, he harvests (removes) $\frac{1}{4}$ of the planted acreage and then plants 1500 more acres. The number of acres of wheat planted approaches what number?

Answers - Practice Problems

- 1.) 3 2.) 5 3.) $\sqrt{6} + 1$ 4.) 2 5.) 4 6.) 7 7.) $\sqrt{2} - 1$ 8.) $\frac{\sqrt{5} + 1}{2}$
- 9.) 2 10.) $\sqrt{2} - 1$ 11.) 6000 acres