

Definition: Given a sequence of numbers $\{a_n\}$, the expression

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

is called an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^n a_k \end{aligned}$$

is called the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L , we say that the series converges and that its **sum** is L . In this case, we also write

$$a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Example 1) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.

Proof: The series is $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$. Consider the sequence $\{s_n\}$ of partial sums.

$$\begin{aligned} s_1 &= \frac{1}{2} & s_4 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5} \\ s_2 &= \frac{1}{2} + \frac{1}{6} = \frac{2}{3} & s_5 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6} \\ s_3 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4} & s_6 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} = \frac{6}{7} \end{aligned}$$

It appears that $s_n = \frac{n}{n+1}$. If that was so, then the sum of the series can be easily found as the limit of the sequence of partial sums:

$$s = \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

This is the case indeed. We can prove it in general using partial fractions:

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

and so

$$\begin{aligned}
 s_1 &= a_1 = \frac{1}{1} - \frac{1}{2} = 1 - \frac{1}{2} \\
 s_2 &= a_1 + a_2 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3} \\
 s_3 &= a_1 + a_2 + a_3 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4} \\
 s_n &= a_1 + a_2 + \dots + a_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}
 \end{aligned}$$

In this case, each partial sum of the series has only a fixed number of terms after cancellation. We call such a series a **telescoping sum**.

Definition: **Geometric series** are of the form

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$

where a and r are fixed real numbers and $a \neq 0$.

Theorem: If $|r| < 1$, the geometric series $a + ar + ar^2 + \dots + ar^{n-1} + \dots$ converges to $\frac{a}{1-r}$ and if $|r| \geq 1$, the series diverges.

Example 2) Determine whether the given series converges or diverges. If it converges, find the sum of the series.

a) $\sum_{n=0}^{\infty} 2^{n-1} \cdot 3^{2-n}$

We start by re-writing a_n

$$a_n = 2^{n-1} \cdot 3^{2-n} = \frac{2^n}{2} \cdot \frac{9}{3^n} = \frac{9}{2} \left(\frac{2}{3}\right)^n$$

We can now determine that $a = \frac{9}{2}$ and $r = \frac{2}{3}$. Since $-1 < r < 1$, the series converges and its sum is

$$s = \frac{a}{1-r} = \frac{\frac{9}{2}}{1 - \frac{2}{3}} = \frac{27}{2}$$

b) $\sum_{n=3}^{\infty} 3^{2n-1} \cdot (-5)^{2-n}$

We start by re-writing a_n

$$a_n = 3^{2n-1} \cdot (-5)^{2-n} = \frac{9^n}{3} \cdot \frac{25}{(-5)^n} = \frac{25}{3} \left(-\frac{9}{5}\right)^n$$

We can now determine that $a = \frac{25}{3}$ and $r = -\frac{9}{5}$. Since $r < -1$, the series diverges and so the sum is not defined. There are more examples in the separate handout dedicated to geometric series.

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \rightarrow 0$.

Proof: Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with sum $s \in \mathbb{R}$. Then $\lim_{n \rightarrow \infty} s_n = s$. Clearly $a_n = s_n - s_{n-1}$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = s - s = 0$$

Theorem: (nth term test) If $\lim_{n \rightarrow \infty} a_n$ doesn't exist or exists but is not zero, then $\sum_{n=1}^{\infty} a_n$ diverges.

The "smallness" of the n th term is a necessary but not sufficient condition of the convergence of the series. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though the n th term, a_n approaches zero. The logical conclusion is that the n th term test can only be used to prove divergence of a series.

Example 3) Prove that $\sum_{n=1}^{\infty} \frac{n-2}{2n+1}$ diverges.

Proof: $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n-2}{2n+1} = \lim_{n \rightarrow \infty} \frac{1 - \frac{2}{n}}{2 + \frac{1}{n}} = \frac{1}{2}$. Since the n th term does not approach zero, the series diverges.

Theorem: If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

- 1) Sum Rule $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
- 2) Difference Rule $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
- 3) Constant Multiple Rule: $\sum k a_n = k \sum a_n = kA$ for any $k \in \mathbb{R}$

These properties can be proved using the properties of limits of sequences. Consequences:

- 1) Every non-zero constant multiple of a divergent series diverges.
- 2) If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverge.

Example 4.) Compute the sum of each of the following series or state if the series diverges.

a) $\sum_{n=0}^{\infty} \frac{7 \cdot 2^n - 3^n}{5^n}$

Solution:

$$\sum_{n=0}^{\infty} \frac{7 \cdot 2^n - 3^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{7 \cdot 2^n}{5^n} - \frac{3^n}{5^n} \right) = \sum_{n=0}^{\infty} 7 \left(\frac{2^n}{5^n} \right) - \sum_{n=0}^{\infty} \left(\frac{3^n}{5^n} \right) = \sum_{n=0}^{\infty} 7 \left(\frac{2}{5} \right)^n - \sum_{n=0}^{\infty} \left(\frac{3}{5} \right)^n$$

We separately compute the sums of the geometric series. In the first series, $a = 7$ and $r = \frac{2}{5}$ and in the second, $a = 1$ and $r = \frac{3}{5}$.

$$s_1 = \frac{7}{1 - \frac{2}{5}} = \frac{35}{3} \quad \text{and} \quad s_2 = \frac{1}{1 - \frac{3}{5}} = \frac{5}{2}$$

and so the difference of the two series is $\frac{35}{3} - \frac{5}{2} = \frac{55}{6}$.

b) $\sum_{n=0}^{\infty} e^n + e^{-n}$

Solution:

$$\sum_{n=0}^{\infty} e^n + e^{-n} = \sum_{n=0}^{\infty} e^n + \sum_{n=0}^{\infty} e^{-n}$$

The first series diverges because it is a geometric series with $r = e > 1$. Thus the sum of the two series diverges as well.

Recall the Monotonic Sequence Theorem: If a non-decreasing sequence is bounded from above, then it is convergent. A consequence of this is the following theorem.

Theorem: A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms converges if and only if its partial sums are bounded from above.

Proof: If the sequence $\{a_n\}$ has only non-negative terms, then the sequence $\{s_n\}$ of partial sums is non-decreasing.

Example 5.) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof: We group the terms into clusters that each add up to a number greater than $\frac{1}{2}$. Since there are infinitely many such clusters, the sequence of partial sums is not bounded from above. Thus the series diverges.

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{2}{4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{4}{8}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right)}_{> \frac{8}{16}} + \dots$$

This is an excellent example to demonstrate that the n th term may approach zero and yet the series diverges.

Theorem: (The Integral Test) Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x) dx$ both converge or both diverge.

Proof. Suppose that the conditions hold. Suppose that $N = 1$. If the function f is decreasing, then all left-hand Riemann sums overestimate the area under the graph and all right-hand sums underestimate the same area. Consider now the following:

$a_1 + a_2 + a_3 + \dots + a_n$ is a left sum for f on the interval $[1, n+1]$ with the partition $\{1, 2, 3, \dots, n+1\}$

$$\int_1^{n+1} f(x) dx \leq a_1 + a_2 + a_3 + \dots + a_n$$

$a_2 + a_3 + \dots + a_n$ is a right sum for f on $[1, n]$ with the partition $\{1, 2, 3, \dots, n\}$. Thus

$$a_2 + a_3 + \dots + a_n \leq \int_1^n f(x) dx$$

Add a_1 to both sides:

$$a_1 + a_2 + a_3 + \dots + a_n \leq a_1 + \int_1^n f(x) dx$$

And so we have that

$$\int_1^{n+1} f(x) dx \leq a_1 + a_2 + a_3 + \dots + a_n \leq a_1 + \int_1^n f(x) dx$$

These inequalities are true for all n , and continue to hold as n approaches infinity.

If $\int_1^\infty f(x) dx$ is finite, then $a_1 + \int_1^\infty f(x) dx$ is an upper limit of the sequence of partial sums, and so $\sum_{n=1}^\infty a_n$ is finite.

If $\int_1^\infty f(x) dx$ is infinite, then $\int_1^{n+1} f(x) dx$ is not bounded from above and so the sequence of partial sums is also not bounded from above, thus $\sum_{n=1}^\infty a_n$ is infinite. Thus the series and the integral are both finite or both infinite.

Example 6.) Determine whether the given series is convergent or not.

a) $\sum_{n=1}^\infty \frac{1}{n}$

This will be the second time we prove that this series diverges, but it is a very famous series. It is called the harmonic series. We will use the integral test. The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ is decreasing and all terms are positive. Also, the function $f(x) = \frac{1}{x}$ is continuous, positive, and decreasing on $[1, \infty)$, so we may apply the integral test.

Since $\int_1^\infty \frac{1}{x} dx$ diverges, so does $\sum_{n=1}^\infty \frac{1}{n}$.

b) $\sum_{n=1}^\infty \frac{1}{n^2}$

We will use the integral test. The sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$ is decreasing and all terms are positive. Also, the function $f(x) = \frac{1}{x^2}$ is continuous, positive, and decreasing on $[1, \infty)$, so we may apply the integral test. Since $\int_1^{\infty} \frac{1}{x^2} dx$ converges, so does $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

c) $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$

We will use the integral test. The sequence $\frac{1}{3 \ln 3}, \frac{1}{4 \ln 4}, \frac{1}{5 \ln 5}, \dots$ is decreasing and all terms are positive. Also, the function $f(x) = \frac{1}{x \ln x}$ is continuous, and positive on $[3, \infty)$. We may apply the integral test, but first we must prove that $f(x) = \frac{1}{x \ln x}$ is decreasing on $[3, \infty)$. We differentiate $f(x)$

$$\frac{d}{dx} \left(\frac{1}{x \ln x} \right) = \frac{d}{dx} \left((x \ln x)^{-1} \right) = -1 (x \ln x)^{-2} (\ln x + 1) = -\frac{1 + \ln x}{(x \ln x)^2}$$

Since the derivative of f is negative on $[3, \infty)$, the function is decreasing and so the conditions for applying the integral test all hold. We compute the improper integral $\int_3^{\infty} \frac{1}{x \ln x} dx$. We use integration by substitution to compute the indefinite integral: let $u = \ln x$ then $du = \frac{1}{x} dx$

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x} \left(\frac{1}{x} dx \right) = \int \frac{1}{u} du = \ln |u| + C = \ln |\ln x| + C$$

Now for the improper integral:

$$\int_3^{\infty} \frac{1}{x \ln x} dx = \lim_{N \rightarrow \infty} \int_3^N \frac{1}{x \ln x} dx = \lim_{N \rightarrow \infty} \ln |\ln x| \Big|_3^N = \lim_{N \rightarrow \infty} (\ln \ln N - \ln \ln 3) = \infty$$

Since the integral diverges, so does the series $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$.

Practice Problems

Determine which of the following series converge and which diverge.

Please note that most of these problems can be solved using several different methods.

$$1. \sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}$$

$$6. \sum_{n=1}^{\infty} \sqrt[n]{n}$$

$$11. \sum_{n=1}^{\infty} \operatorname{sech} n$$

$$2. \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

$$7. \sum_{n=1}^{\infty} \frac{8}{n^2}$$

$$12. \sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$$

$$3. \sum_{n=0}^{\infty} \frac{1}{n^2+1}$$

$$8. \sum_{n=2}^{\infty} \frac{\ln n}{n}$$

$$13. \sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$4. \sum_{n=0}^{\infty} \frac{5^n + 3^n}{4^n}$$

$$9. \sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

$$14. \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

$$5. \sum_{n=0}^{\infty} \sqrt{n}$$

$$10. \sum_{n=1}^{\infty} \frac{2}{e^n}$$

Answers - Practice Problems

- 1.) converges to $\frac{3}{4}$ 2.) converges to 1 3.) converges 4.) diverges 5.) diverges 6.) diverges
 7.) converges 8.) diverges 9.) converges 10.) converges 11.) converges 12.) diverges
 13.) converges 14.) diverges

Solutions

1. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}$ converges to $\frac{3}{4}$, it is a telescoping sum

$$a_n = \frac{1}{n^2 + 2n} = \frac{1}{n(n+2)} = \frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

$$s_1 = a_1 = \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right) = \frac{1}{3}$$

$$s_2 = a_1 + a_2 = \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) \right] = \frac{11}{24}$$

$$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) \right] = \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} \right) = \frac{21}{40}$$

$$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) \right] = \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6} \right) = \frac{17}{30}$$

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n \\ &= \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n+1} \right) + \left(\frac{1}{n} - \frac{1}{n+2} \right) \right] \\ &= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{1}{2} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \end{aligned}$$

Thus $s_n = \frac{1}{2} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right)$ and so

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{3}{4}$$

2. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$ converges to 1

Solution: We use partial fractions to decompose $a_n = \frac{2n+1}{n^2(n+1)^2}$

$$\begin{aligned} \frac{2n+1}{n^2(n+1)^2} &= \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2} \\ \frac{2n+1}{n^2(n+1)^2} &= \frac{An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2}{n^2(n+1)^2} \\ 2n+1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 \end{aligned}$$

Let $n = 0$

$$\begin{aligned} 2n + 1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 \quad \text{becomes} \\ 1 &= B \end{aligned}$$

Let $n = -1$

$$\begin{aligned} 2n + 1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 \quad \text{becomes} \\ -1 &= D \end{aligned}$$

Let $n = 1$

$$\begin{aligned} 2n + 1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 \quad \text{becomes} \\ 3 &= 4A + 4 + 2C - 1 \\ 0 &= 4A - 2C \\ C &= 2A \end{aligned}$$

Let $n = 2$

$$\begin{aligned} 2n + 1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 \quad \text{becomes} \\ 5 &= 18A + 9 + 12C - 4 \\ 0 &= 18A + 12C \\ 0 &= 3A + 2C \end{aligned}$$

So we finally have this system:

$$\begin{aligned} C &= 2A \\ 3A + 2C &= 0 \end{aligned}$$

we solve the system and obtain $A = C = 0$. Thus the decomposition of a_n is as follows:

$$a_n = \frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

Thus the sequence of partial sums is as follows:

$$\begin{aligned} s_1 &= a_1 = \frac{1}{1^2} - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4} \\ s_2 &= a_1 + a_2 = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) = 1 - \frac{1}{9} = \frac{8}{9} \\ s_3 &= a_1 + a_2 + a_3 = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) = 1 - \frac{1}{16} = \frac{15}{16} \\ s_n &= a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n \\ &= \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \dots + \left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right) + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \\ &= 1 - \frac{1}{(n+1)^2} \end{aligned}$$

$$3. \sum_{n=0}^{\infty} \frac{1}{n^2+1} \text{ converges by integral test}$$

4. $\sum_{n=0}^{\infty} \frac{5^n + 3^n}{4^n}$ diverges because

$$\sum_{n=0}^{\infty} \frac{5^n + 3^n}{4^n} = \sum_{n=0}^{\infty} \frac{5^n}{4^n} + \sum_{n=0}^{\infty} \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$$

and the first geometric series diverges

5. $\sum_{n=0}^{\infty} \sqrt{n}$ diverges since a_n does not approach 0

6. $\sum_{n=1}^{\infty} \sqrt[n]{n}$ diverges since a_n does not approach 0

7. $\sum_{n=1}^{\infty} \frac{8}{n^2}$ converges since it is a constant times $\sum_{n=1}^{\infty} \frac{1}{n^2}$

8. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges by the integral test:

The function $f(x) = \frac{\ln x}{x}$ is continuous and positive on $[2, \infty)$. We may apply the integral test if we also show that f is decreasing there. $\frac{d}{dx} \left(\frac{\ln x}{x} \right) = \frac{1}{x^2} (1 - \ln x)$ which is negative on $[3, \infty)$. So after $n = 3$, the conditions hold. $\int \frac{\ln x}{x} dx = \frac{1}{2} (\ln x)^2 + C$. The improper integral $\int_3^{\infty} \frac{\ln x}{x} dx$ diverges and therefore so does the series.

9. $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges by the integral test

10. $\sum_{n=1}^{\infty} \frac{2}{e^n}$ converges, a geometric sequence with $a = \frac{2}{e}$ and $r = \frac{1}{e}$ with $0 < \frac{1}{e} < 1$

11. $\sum_{n=1}^{\infty} \operatorname{sech} n$ converges by the integral test

$$\frac{d}{dx} \left(\frac{2}{e^x + e^{-x}} \right) = 2 \frac{d}{dx} (e^x + e^{-x})^{-1} = 2(-1)(e^x + e^{-x})^{-2} (e^x - e^{-x})$$

$$\int \frac{2}{e^x + e^{-x}} dx = 2 \arctan(e^x) + C \quad \int_0^{\infty} \frac{2}{e^x + e^{-x}} dx = \frac{\pi}{2}$$

12. $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ diverges

We decompose a_n using partial fractions:

$$\frac{n+2}{n^2-n} = \frac{3}{n-1} - \frac{2}{n}$$

Then s_n becomes $s_n = 2 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \right)$

$$\begin{aligned}
 s_n &= \frac{3}{1} - \underbrace{\frac{2}{2} + \frac{3}{2}} - \underbrace{\frac{2}{3} + \frac{3}{3}} - \underbrace{\frac{2}{4} + \frac{3}{4}} - \frac{2}{5} + \dots - \underbrace{\frac{2}{n-1} + \frac{3}{n-1}} - \frac{2}{n} \\
 &= 3 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n-1} \right) - \frac{2}{n}
 \end{aligned}$$

This shows that the sequence of partial sums is not bounded from above, thus the series diverges.

13. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the integral test

14. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the integral test