

Theorem: (The Alternating Series Test or Leibniz's Test) The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$$

converges if all three of the following conditions are satisfied:

- 1) u_n is positive for all n .
- 2) u_n are (eventually) non-increasing: $u_n \geq u_{n+1}$ for all $n \geq N$ for some integer N .
- 3) $\lim_{n \rightarrow \infty} u_n = 0$

Proof: Assume that $N = 1$. If n is even, say $n = 2k$ then the partial sum s_n is

$$s_n = (u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2k-1} - u_{2k})$$

Since $\{u_n\}$ is non-increasing, each difference $u_j - u_{j+1}$ is non-negative, thus s_n is non-negative and non-decreasing. Let us look at s_n again, but this time as

$$s_n = u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots - (u_{2k-2} - u_{2k-1}) - u_{2k}$$

This line shows that $s_n \leq u_1$. Thus the sequence $\{s_n\} = \{s_{2k}\}$ of even partial sums is bounded from above (one upper bound is u_1). Because $\{s_n\} = \{s_{2k}\}$ is a sequence that is non-decreasing and bounded from above, it is convergent. Let us denote the limit by L .

Let us now consider the odd partial sums $\{s_{2k+1}\} = s_{2k} + u_{2k+1}$. Recall that $\lim_{n \rightarrow \infty} u_n = 0$. Then

$$\lim_{n \rightarrow \infty} s_{2k+1} = \lim_{n \rightarrow \infty} s_{2k} + u_{2k+1} = \lim_{n \rightarrow \infty} s_{2k} + \lim_{n \rightarrow \infty} u_{2k+1} = L + 0 = L$$

This completes our proof.

Theorem: (The Alternating Series Estimation Theorem) If an alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ satisfies Leibniz's test, then

$$s_n = u_1 - u_2 + u_3 - u_4 + \dots + (-1)^{n+1} u_n$$

approximates the sum S of the series with an error whose absolute value is less than u_{n+1} , the absolute value of the first unused term. Furthermore, the sum S lies between any two successive partial sums s_n and s_{n+1} , and the remainder, $S - s_n$ has the same sign as the first unused term.

Definition: A series $\sum a_n$ **converges absolutely** if the corresponding series of absolute values $\sum |a_n|$ converges.

Definition: A series that converges but does not converge absolutely **converges conditionally**.

Theorem: (The Absolute Convergence Test) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof: Suppose that $\sum_{n=1}^{\infty} |a_n|$ converges. Then the series $\sum_{n=1}^{\infty} 2|a_n|$ also converges. Consider now the series

$\sum_{n=1}^{\infty} (|a_n| + a_n)$. We claim that this series converges as well, by the comparison test: for all $n \in \mathbb{N}$,

$$0 \leq |a_n| + a_n \leq 2|a_n|$$

So $\sum_{n=1}^{\infty} (|a_n| + a_n)$ also converges. Now consider the difference $\sum_{n=1}^{\infty} (|a_n| + a_n) - |a_n|$. Since this is the difference of two convergent series, it also converges. This completes our proof.

Theorem: (The Rearrangement Theorem for Absolutely Convergent Series) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and

$b_1, b_2, b_3, \dots, b_n, \dots$ is any rearrangement of the sequence $\{a_n\}$ then $\sum_{n=1}^{\infty} b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$$

Note that this is not true for conditionally convergent series.

Sample Problems

1. Determine convergence or divergence of the alternative series.

a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

d) $\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$

g) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 + 5}{n^2 + 4}$

e) $\sum_{n=1}^{\infty} (-1)^n \ln \left(1 + \frac{1}{n} \right)$

h) $\sum_{n=1}^{\infty} (-2)^{-n}$

c) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 - 1}$

f) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\tan^{-1} n}{n^2 + 1}$

2. Determine whether the following series converge absolutely, converge conditionally, or diverge.

a) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$

d) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$

f) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

e) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (n!)^2}{(2n)!}$

g) $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})$

c) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$

f) $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2 3^n}{(2n+1)!}$

Solutions - Sample Problems

1. Determine convergence or divergence of the alternative series.

a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

converges since $\{a_n\}$ is alternating, decreasing, and approaches zero

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 + 5}{n^2 + 4}$

diverges since a_n fails to approach zero

c) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 - 1}$

converges since $\{a_n\}$ is alternating, decreasing, and approaches zero

d) $\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$

converges since $\{a_n\}$ is alternating, decreasing after $N = 11$, and approaches zero

e) $\sum_{n=1}^{\infty} (-1)^n \ln \left(1 + \frac{1}{n} \right)$

converges since $\{a_n\}$ is alternating, decreasing, and approaches zero

f) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\tan^{-1} n}{n^2 + 1}$

converges since $\{a_n\}$ is alternating, decreasing, and approaches zero

g) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$ diverges since a_n fails to approach zero

h) $\sum_{n=1}^{\infty} (-2)^{-n}$ converges since it is a geometric series with $r = -\frac{1}{2}$

2. Determine whether the following series converge absolutely, converge conditionally, or diverge.

a) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$

The series converges absolutely. We prove it using the ratio test

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{100^{n+1}}{(n+1)!}}{\frac{100^n}{n!}} = \lim_{n \rightarrow \infty} \left(\frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} \right) = \lim_{n \rightarrow \infty} \left(\frac{100 \cdot 100^n}{(n+1)n!} \cdot \frac{n!}{100^n} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{100}{n+1} \right) = 0 < 1 \end{aligned}$$

$$b) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$$

This series converges conditionally. $\frac{1}{\sqrt{n}}$ is decreasing and approaches zero. However, it is not absolutely convergent, because $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the integral test.

$$c) \sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$$

This series converges absolutely because $\sum_{n=1}^{\infty} \left| \frac{\cos n\pi}{n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges by the integral test

$$d) \sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$$

This series converges conditionally because it satisfies Leibniz's test. However, the series $\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the integral test

$$e) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (n!)^2}{(2n)!}$$

This series converges absolutely because $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1} (n!)^2}{(2n)!} \right| = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges by the ratio test

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n \rightarrow \infty} \frac{\frac{((n+1)n!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n \rightarrow \infty} \frac{(n+1)^2 (n!)^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{(n!)^2} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{2(n+1)(2n+1)} = \lim_{n \rightarrow \infty} \frac{n+1}{2(2n+1)} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{2\left(2 + \frac{1}{n}\right)} = \frac{1}{4} < 1 \end{aligned}$$

$$f) \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2 3^n}{(2n+1)!} \quad \text{This series converges absolutely because of the ratio test}$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{((n+1)!)^2 3^{n+1}}{(2(n+1)+1)!}}{\frac{(n!)^2 3^n}{(2n+1)!}} = \lim_{n \rightarrow \infty} \frac{((n+1)n!)^2 3^n \cdot 3}{(2n+3)!} \cdot \frac{(2n+1)!}{(n!)^2 3^n} \\ &= \lim_{n \rightarrow \infty} \frac{3(n+1)^2 (n!)^2 (2n+1)!}{(2n+3)(2n+2)(2n+1)!(n!)^2} = \lim_{n \rightarrow \infty} \frac{3(n+1)^2}{(2n+3)(2n+2)} = \lim_{n \rightarrow \infty} \frac{3(n+1)^2}{(2n+3)2(n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{3(n+1)}{2(2n+3)} = \frac{3}{4} < 1 \end{aligned}$$

$$f) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1} \quad \text{This series diverges because } a_n \text{ fails to approach zero.}$$

$$g) \sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})$$

This series converges conditionally, because it satisfies Leibniz's test:

$$\begin{aligned} \lim_{n \rightarrow \infty} |a_n| &= \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n \rightarrow \infty} \left((\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right) = \lim_{n \rightarrow \infty} \left(\frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0 \end{aligned}$$

However, the series $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ diverges by the comparison test:

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \geq \frac{1}{\sqrt{n+1} + \sqrt{n+1}} = \frac{1}{2\sqrt{n}}$$

and $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ is divergent.

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