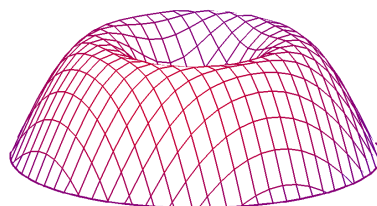
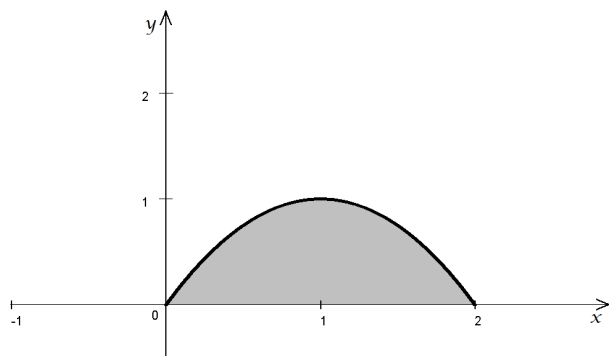
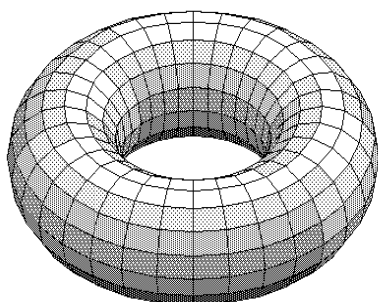


Sample Problems

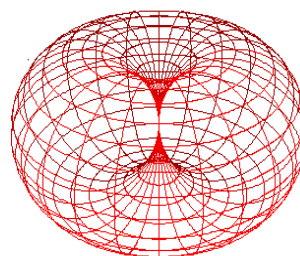
1. Consider the region bounded by $f(x) = -x^2 + 2x$, $y = 0$, and $x = 2$. Compute the volume of the object we obtain when rotating this region about the y -axis.



2. Compute the volume of the object that we obtain by rotating the region between $y = \frac{1}{3}x$, $y = 0$, and $x = 6$
 a) about the y -axis b) about the line $x = -1$
3. Compute the volume of the object that we obtain by rotating the region between $y = \sin x$, $y = 0$, and $x = \pi$ about the y -axis.
4. (the torus) Consider the region bounded by the circle $(x - 2)^2 + y^2 = 1$. Compute the volume of the object we obtain when rotating the region



a)



b)

- a) about the y -axis
 b) about the line $x = 1$
 c) Assume that $R > r$. A circle is centered at $(R, 0)$ and has radius r . We rotate it about the y -axis. Compute the volume of this torus.

Practice Problems

1. Compute the volume of the object we obtain by rotating the region bounded by $y = x^2$, $y = 0$, and $x = 1$ about the y -axis.
2. Compute the volume of the object we obtain by rotating the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ about the y -axis.
3. Compute the volume of the object we obtain by rotating the region bounded by $y = \frac{1}{x}$, $y = 0$, $x = 1$ and $x = 2$, about the y -axis.
4. a) The torus created by rotating the unit circle centered at $(1, 0)$ about the y -axis.
b) The torus created by rotating the unit circle centered at $(2, 0)$ about the y -axis.

Answers - Sample Problems

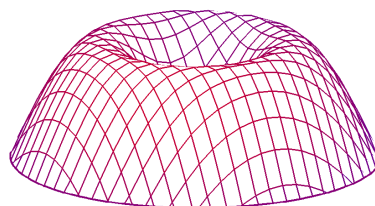
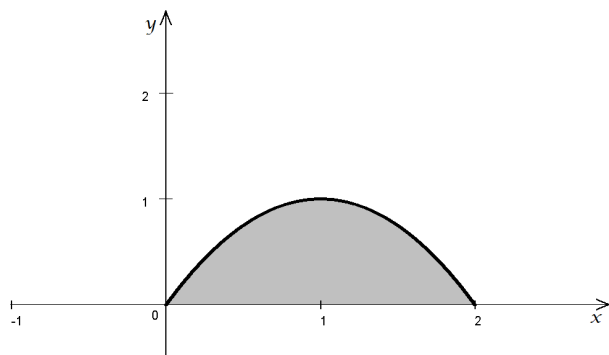
- 1.) $\frac{8}{3}\pi$ 2.) a) 48π b) 60π 3.) $2\pi^2$ 4.) a) $4\pi^2$ b) $2\pi^2$ c) $V = 2\pi^2 r^2 R$

Answers - Practice Problems

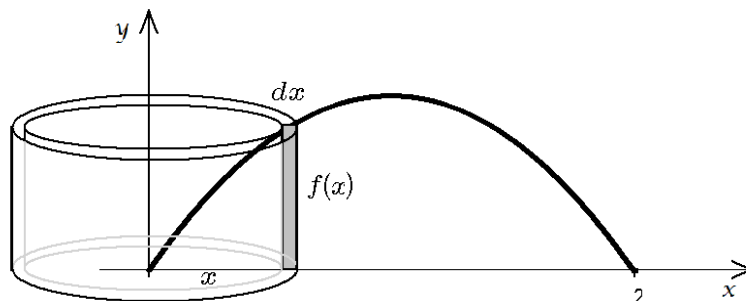
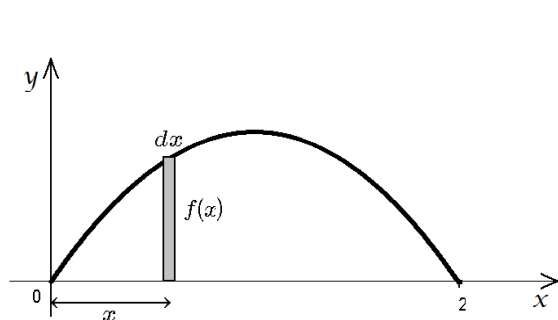
- 1.) $\frac{\pi}{2}$ 2.) $\frac{128}{5}\pi$ 3.) 2π 4.) a) $2\pi^2$ b) $4\pi^2$

Solutions of Sample Problems

1. Consider the region bounded by $f(x) = -x^2 + 2x$, $y = 0$, and $x = 2$. Compute the volume of the object we obtain when rotating this region about the y -axis.



Solution: We partition the interval $[0, 2]$ into very small intervals over which we approximate the function to be constant. The area under the graph becomes a sum of many rectangles. Consider rotating one such rectangle about the y -axis. The object we obtain is called a cylindrical shell.



The volume of such a cylindrical shell is approximated by

$$V_i = \text{circumference} \cdot \text{height} \cdot \text{thickness}$$

In this particular case, the circumference is $2\pi x$, the height is $f(x)$, and the thickness is dx . The total volume is then

$$\begin{aligned} V &= \int_0^2 (2\pi x) f(x) \, dx = \int_0^2 2\pi x (-x^2 + 2x) \, dx = 2\pi \int_0^2 -x^3 + 2x^2 \, dx = 2\pi \left(\frac{-x^4}{4} + \frac{2x^3}{3} \right) \Big|_0^2 \\ &= 2\pi \left(\frac{-16}{4} + \frac{16}{3} \right) = 32\pi \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{32}{12}\pi = \boxed{\frac{8}{3}\pi} \end{aligned}$$

2. Compute the volume of the object that we obtain by rotating the region between $y = \frac{1}{3}x$, $y = 0$, and $x = 6$

a) about the y -axis

Solution:

$$V = \int_0^6 (2\pi x) f(x) \, dx = \int_0^6 2\pi x \left(\frac{1}{3}x \right) \, dx = \frac{2\pi}{3} \int_0^6 x^2 \, dx = \frac{2\pi}{3} \left(\frac{x^3}{3} \right) \Big|_0^6 = \frac{2\pi}{9} (6^3 - 0) = \boxed{48\pi}$$

b) about the line $x = -1$

Solution: If we rotate around the line $x = -1$, then the cylindrical shell at x will have a radius of $x + 1$.

$$\begin{aligned} V &= \int_0^6 2\pi (x+1) f(x) dx = \int_0^6 2\pi (x+1) \left(\frac{1}{3}x\right) dx = \frac{2\pi}{3} \int_0^6 x^2 + x dx = \frac{2\pi}{3} \left(\frac{x^3}{3} + \frac{x^2}{2}\right) \Big|_0^6 \\ &= \frac{2\pi}{3} \left[\left(\frac{6^3}{3} + \frac{6^2}{2}\right) - 0\right] = \frac{2\pi}{3} (90) = \boxed{60\pi} \end{aligned}$$

3. Compute the volume of the object that we obtain by rotating the region between $y = \sin x$, $y = 0$, and $x = \pi$ about the y -axis.

Solution:

$$V = \int_0^\pi 2\pi (x) f(x) dx = \int_0^\pi 2\pi x (\sin x) dx = 2\pi \int_0^\pi x \sin x dx$$

We will first evaluate the indefinite integral by integrating by parts. We will use the formula $\int f'g = fg - \int fg'$.

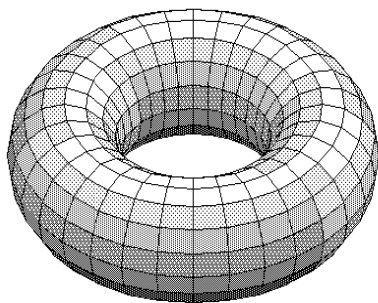
Let $g(x) = x$ and $f'(x) = \sin x$. Then also $g'(x) = 1$ and $f(x) = -\cos x$ and so

$$\begin{aligned} \int f'g &= fg - \int fg' \text{ becomes} \\ \int x \sin x dx &= x(-\cos x) - \int -\cos x dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C \end{aligned}$$

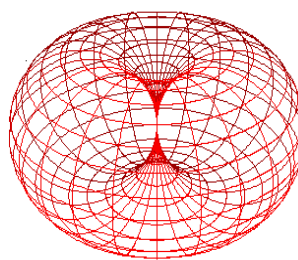
We are now ready to compute the definite integral

$$\begin{aligned} V &= 2\pi \int_0^\pi x \sin x dx = 2\pi (-x \cos x + \sin x) \Big|_0^\pi = 2\pi [(-\pi \cos \pi + \sin \pi) - (-0 \cos 0 + \sin 0)] \\ &= 2\pi [(-\pi(-1) + 0) - (0)] = \boxed{2\pi^2} \end{aligned}$$

4. (the torus) Consider the region bounded by the circle $(x-2)^2 + y^2 = 1$. Compute the volume of the object we obtain when rotating the region



a)



b)

a) about the y -axis

Solution: We solve for y in the equation and obtain $y = \pm\sqrt{1 - (x - 2)^2}$. We will rotate the region between the x -axis and $f(x) = \sqrt{1 - (x - 2)^2}$ (which is the upper half) and then multiply the volume by 2.

$$V = 2 \int_1^3 2\pi x f(x) dx = 4\pi \int_1^3 x \sqrt{1 - (x - 2)^2} dx$$

Let us first substitute $u = x - 2$. Then clearly $x = u + 2$ and $dx = du$. We will need to substitute the limits of the integral as well. When $x = 1$, then $u = 1 - 2 = -1$ and when $x = 3$, then $u = 3 - 2 = 1$. Now the volume is

$$\begin{aligned} V &= 4\pi \int_1^3 x \sqrt{1 - (x - 2)^2} dx = 4\pi \int_{-1}^1 (u + 2) \sqrt{1 - u^2} du = 4\pi \int_{-1}^1 u \sqrt{1 - u^2} + 2\sqrt{1 - u^2} du \\ &= 4\pi \left(\int_{-1}^1 u \sqrt{1 - u^2} du + \int_{-1}^1 2\sqrt{1 - u^2} du \right) \end{aligned}$$

The two integrals can be computed via different methods. The first integral, $\int_{-1}^1 u \sqrt{1 - u^2} du$ is zero because the integrand is an odd function and the limits are opposites. (Computation will yield for the same result, just substitute $m = u^2$.) The second integral, $\int_{-1}^1 2\sqrt{1 - u^2} du = 2 \int_{-1}^1 \sqrt{1 - u^2} du$ is $\pi(1)^2 = \pi$ because it expresses the area of the unit circle. (Recall that if we replace $f(x)$ by y in $f(x) = \sqrt{1 - x^2}$ and square both sides, we get $x^2 + y^2 = 1$.) The computation for $\int_{-1}^1 \sqrt{1 - u^2} du = \frac{\pi}{2}$ can be found in the handout Integrating by Substitution. Thus the volume is

$$V = 4\pi \left(\int_{-1}^1 u \sqrt{1 - u^2} du + \int_{-1}^1 2\sqrt{1 - u^2} du \right) = 4\pi(0 + \pi) = \boxed{4\pi^2}$$

b) about the line $x = 1$

Solution:

$$V = 2 \int_1^3 (2\pi)(x - 1) f(x) dx = 4\pi \int_1^3 (x - 1) \sqrt{1 - (x - 2)^2} dx$$

Again we will substitute $u = x - 2$. As before, $x = u + 2$ and $dx = du$. We will need to substitute the limits of the integral as well. When $x = 1$, then $u = 1 - 2 = -1$ and when $x = 3$, then $u = 3 - 2 = 1$. Now the volume is

$$\begin{aligned} V &= 4\pi \int_1^3 (u + 2 - 1) \sqrt{1 - u^2} du = 4\pi \int_{-1}^1 (u + 1) \sqrt{1 - u^2} du = 4\pi \left(\int_{-1}^1 u \sqrt{1 - u^2} + \sqrt{1 - u^2} du \right) \\ &= 4\pi \left(\int_{-1}^1 u \sqrt{1 - u^2} du + \int_{-1}^1 \sqrt{1 - u^2} du \right) \end{aligned}$$

The first integral is zero because the integrand is odd and the limits are opposites. The second integral $\int_{-1}^1 \sqrt{1-u^2} du$ is $\frac{\pi}{2}$ because $\int_{-1}^1 \sqrt{1-u^2} du$ is the area under a semicircle of radius 1, which is $\frac{\pi}{2}$. (The computation for $\int_{-1}^1 \sqrt{1-u^2} du = \frac{\pi}{2}$ can be found in the handout Integrating by Substitution.) Putting all these together,

$$V = 4\pi \left(\int_{-1}^1 u\sqrt{1-u^2} du + \int_{-1}^1 \sqrt{1-u^2} du \right) = 4\pi \left(0 + \frac{\pi}{2} \right) = \boxed{2\pi^2}$$

c) Assume that $R > r$. A circle is centered at $(R, 0)$ and has radius r . We rotate it about the y -axis. Compute the volume of this torus.

Solution: We will use shells to compute the volume we obtain when we rotate the upper semicircle. We get the right volume if we just double that.

$$V = 2 \int_{R-r}^{R+r} 2\pi x \sqrt{r^2 - (x-R)^2} dx = 4\pi \int_{R-r}^{R+r} x \sqrt{r^2 - (x-R)^2} dx$$

Let $u = x - R$. Then of course $du = dx$ and $x = u + R$. We will substitute the limits as well.

$$\begin{aligned} V &= 4\pi \int_{R-r}^{R+r} x \sqrt{r^2 - (x-R)^2} dx = 4\pi \int_{-r}^r (u+R) \sqrt{r^2 - u^2} du = 4\pi \int_{-r}^r u \sqrt{r^2 - u^2} du + R \int_{-r}^r \sqrt{r^2 - u^2} du \\ &= 4\pi \left(\int_{-r}^r u \sqrt{r^2 - u^2} du + \int_{-r}^r R \sqrt{r^2 - u^2} du \right) \end{aligned}$$

The two integrals are substantially different. The first one is zero since the function $u\sqrt{r^2 - u^2}$ is odd and the limits are opposites. The second integral is

$$\int_{-r}^r R \sqrt{r^2 - u^2} du = R \int_{-r}^r \sqrt{r^2 - u^2} du = R \left(\frac{1}{2} \pi r^2 \right)$$

because the integral now expresses the area under a semicircle with radius r . (The computation for $\int_{-r}^r \sqrt{r^2 - u^2} du = \frac{1}{2} \pi r^2$ can be found in the handout Integrating by Substitution.) The volume is

$$V = 4\pi \left(\int_{-r}^r u \sqrt{r^2 - u^2} du + \int_{-r}^r R \sqrt{r^2 - u^2} du \right) = 4\pi \left(0 + \frac{1}{2} \pi r^2 R \right) = \boxed{2\pi^2 r^2 R}$$

Notice that we rotated a circle of radius r in a circular motion with radius R . It is interesting that according to this formula, we can think of the result as multiplying the area of the small circle by the perimeter of the big one: $(\pi r^2)(2\pi R) = 2\pi^2 r^2 R$.