

Definition: Let f be a function with derivatives of orders $1, 2, \dots, n$ in some interval containing a as an interior point. Then the **Taylor polynomial of order n generated by f at $x = a$** is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example 1. Find the Taylor polynomial of order 6 generated by $f(x) = e^x$ at $x = 0$.

$$P_6(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

Definition: Let f be a function with derivatives of all orders throughout an interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

The **Maclaurin series generated by f** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

the Taylor series generated by f at $x = 0$.

Example 2. Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Where does the series converge to $\frac{1}{x}$?

$$\begin{aligned} f(x) &= x^{-1} & f'(x) &= -x^{-2} & f''(x) &= 2x^{-3} & f^{(3)}(x) &= -6x^{-4} & f^{(n)}(x) &= (-1)^n n! x^{-(n+1)} \\ f(2) &= 2^{-1} = \frac{1}{2} & f'(2) &= -2^{-2} = -\frac{1}{4} & f''(2) &= 2 \cdot 2^{-3} = \frac{2}{8} = \frac{1}{4} & f^{(3)}(2) &= -6 \cdot 2^{-4} = -\frac{3}{8} \end{aligned}$$

The Taylor series is

$$\begin{aligned} & f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \dots + \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 + \dots + (-1)^n \frac{1}{2^{n+1}}(x-2)^n \end{aligned}$$

Since this is a geometrical series with first term $\frac{1}{2}$ and common ratio $r = -\frac{x-2}{2}$, it converges absolutely for

$$\left| \frac{x-2}{2} \right| < 1 \text{ and its sum is } \frac{\frac{1}{2}}{1 + \frac{x-2}{2}} = \frac{1}{2+x-2} = \frac{1}{x}.$$

It is easy to determine radius of convergence but unfortunately, when a Taylor series generated by f converges, it may or may not converge to f . One example is the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-1/x^2} & \text{if } x \neq 0 \end{cases}$$

This function has derivatives of all orders at $x = 0$ and $f^{(n)}(x) = 0$ for all n . Thus the Taylor series at $x = 0$ is

$$0 + 0 \cdot x + 0 \cdot x^2 + \dots + 0 \cdot x^n + \dots$$

which is clearly convergent for all x but does not converge to $f(x)$.

Theorem: (Taylor's Theorem) Suppose that $f, f', f'', \dots, f^{(n)}$ are continuous on an open interval I containing a . Then for all positive integer n and all x in I

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

$$\text{where } R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x.$$

If $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ converges to f on I and write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Example 3. Find all values of x for which the Taylor series generated by $f(x) = e^x$ converges to its generating function.

$$\text{By Taylor's theorem, } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + R_n(x)$$

where the remainder term is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \quad \text{for some } c \text{ between } 0 \text{ and } x$$

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}$$

$$\lim_{n \rightarrow \infty} R_n(x) = \lim_{n \rightarrow \infty} \frac{e^c}{(n+1)!}x^{n+1} = e^c \lim_{n \rightarrow \infty} \frac{x^{n+1}}{(n+1)!} = 0$$

and so the Taylor series converges to e^x for all x .

Important Taylor series:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots && \text{on } \mathbb{R} \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots && \text{on } \mathbb{R} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots && \text{on } \mathbb{R} \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \dots && \text{on } (-1, 1) \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 + \dots && \text{on } (-1, 1) \\ \ln(x+1) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots && \text{on } (-1, 1] \\ \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots && \text{on } [-1, 1] \end{aligned}$$

Sample Problems

1. Find the Taylor series generated by each of the following functions.

a) $f(x) = x \sin x$

b) $f(x) = e^{3x}$

c) $f(x) = x^2 \tan^{-1} x$

d) $f(x) = \sinh x$

2. Use a Taylor series to determine the sum of each of the following series.

a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

b) $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$

c) $\frac{\pi}{3} - \frac{\pi^3}{3^3 \cdot 3!} + \frac{\pi^5}{3^5 \cdot 5!} - \frac{\pi^7}{3^7 \cdot 7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{3^{2n+1} (2n+1)!}$

d) $-1 + 2x - 3x^2 + 4x^3 - 5x^4 - \dots = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$

e) $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n+1}$

3. Use the Taylor series generated by $f(x) = x \tan^{-1} x$ to determine the value of $f^{(208)}(0)$.

4. Consider the Taylor series generated by $f(x) = e^x$.

a) Let i be the complex number with $i^2 = -1$. Simplify the expressions $i^2, i^3, i^4, i^5, i^6, \dots$

b) Write the Taylor series generated by $f(x) = e^x$ and evaluate it at $x = i\theta$ to prove that $e^{i\theta} = \cos \theta + i \sin \theta$

c) Use the result $e^{i\theta} = \cos \theta + i \sin \theta$ to compute $e^{i\pi}$

Solutions - Sample Problems

1. Find the Taylor series generated by each of the following functions.

a) $f(x) = x \sin x$

Solution: We take the Taylor series generated by $\sin x$ and multiply it by x .

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots && \text{on } \mathbb{R} \\ x \sin x &= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots && \text{on } \mathbb{R}\end{aligned}$$

b) $f(x) = e^{3x}$

Solution: We take the Taylor series generated by e^x and substitute $3x$ into it.

$$\begin{aligned}e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} && \text{on } \mathbb{R} \\ e^{3x} &= 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} && \text{on } \mathbb{R} \\ &= 1 + 3x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4 + \dots = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n && \text{on } \mathbb{R}\end{aligned}$$

c) $f(x) = x^2 \tan^{-1} x$

Solution: We take the Taylor series generated by $f(x) = \tan^{-1} x$ and multiply it by x^2 .

$$\begin{aligned}\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} && \text{on } [-1, 1] \\ x^2 \tan^{-1} x &= x^3 - \frac{x^5}{3} + \frac{x^7}{5} - \frac{x^9}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1} && \text{on } [-1, 1]\end{aligned}$$

d) $f(x) = \sinh x$

Solution: We will find the Taylor series directly from the derivatives $f^{(n)}(0)$. Recall that $a_n = \frac{f^{(n)}(0)}{n!}$

$$\begin{aligned}f(x) &= \sinh x &\implies f(0) = 0 &\implies a_0 = \frac{f(0)}{0!} = \frac{0}{1} = 0 \\ f'(x) &= \cosh x &\implies f'(0) = 1 &\implies a_1 = \frac{f'(0)}{1!} = \frac{1}{1} = 1 \\ f''(x) &= \sinh x &\implies f''(0) = 0 &\implies a_2 = \frac{f''(0)}{2!} = \frac{0}{2} = 0 \\ f^{(3)}(x) &= \cosh x &\implies f^{(3)}(0) = 1 &\implies a_3 = \frac{f^{(3)}(0)}{3!} = \frac{1}{6} \\ f^{(4)}(x) &= \sinh x &\implies f^{(4)}(0) = 0 &\implies a_4 = \frac{f^{(4)}(0)}{4!} = \frac{0}{24} = 0\end{aligned}$$

and so on, as the derivatives repeat, we have an easy pattern, and so the Taylor series for $\sinh x$ is

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

2. Use a Taylor series to determine the sum of each of the following series.

$$\text{a) } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

Solution: We start with the Taylor series for $\ln(1+x)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \quad \text{on } (-1, 1]$$

Let $x = 1$

$$\begin{aligned} \ln(1+1) &= 1 - \frac{1^2}{2} + \frac{1^3}{3} - \frac{1^4}{4} + \frac{1^5}{5} - \dots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \end{aligned}$$

Thus the sum of the series is $\ln 2$.

$$\text{b) } 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$$

Solution: We start by the Taylor series generated by e^x and evaluate it at $x = -1$.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{on } \mathbb{R} \\ e^{-1} &= 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \end{aligned}$$

Thus the sum of the series is $\frac{1}{e}$

$$\text{c) } \frac{\pi}{3} - \frac{\pi^3}{3^3 \cdot 3!} + \frac{\pi^5}{3^5 \cdot 5!} - \frac{\pi^7}{3^7 \cdot 7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{3^{2n+1} (2n+1)!}$$

Solution: We start by the Taylor series generated by $\sin x$ and evaluate it at $x = \frac{\pi}{3}$.

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} \quad \text{on } \mathbb{R} \\ \sin \frac{\pi}{3} &= \frac{\pi}{3} - \frac{\left(\frac{\pi}{3}\right)^3}{3!} + \frac{\left(\frac{\pi}{3}\right)^5}{5!} - \frac{\left(\frac{\pi}{3}\right)^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{3^{2n+1} (2n+1)!} \end{aligned}$$

Thus the sum of the series is $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$$\text{d) } -1 + 2x - 3x^2 + 4x^3 - 5x^4 \dots = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

Solution: Notice that if we integrate the series term by term, we get

$$-x + x^2 - x^3 + x^4 - x^5 \dots = \sum_{n=1}^{\infty} (-1)^n x^n$$

This is a geometric series with first element $-x$ and common ratio $-x$. Thus this series converges if $|x| < 1$ to the sum $\frac{-x}{1 - (-x)} = -\frac{x}{1+x}$.

$$-x + x^2 - x^3 + x^4 - x^5 \dots = -\frac{x}{1+x} \quad \text{on } (-1, 1)$$

Differentiating both sides of this we obtain

$$-1 + 2x - 3x^2 + 4x^3 - 5x^4 \dots = -\frac{1}{(x+1)^2} \quad \text{on } (-1, 1)$$

and so the sum is $-\frac{1}{(x+1)^2}$ for all $-1 < x < 1$.

$$\text{e) } 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n+1}$$

Solution: Notice that when we multiply this series by x and differentiate it, we get

$$\begin{aligned} f(x) &= 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n+1} \\ xf(x) &= x \left(1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots \right) = x \left(\sum_{n=0}^{\infty} \frac{x^n}{n+1} \right) \\ xf(x) &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\ \frac{d}{dx}(xf(x)) &= 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \\ \frac{d}{dx}(xf(x)) &= \frac{1}{1-x} \quad \text{on } (-1, 1) \\ xf(x) &= \int \frac{1}{1-x} dx \quad \text{on } (-1, 1) \\ xf(x) &= -\ln(1-x) \quad \text{on } (-1, 1) \\ f(x) &= -\frac{1}{x} \ln(1-x) \quad \text{on } (-1, 1) \end{aligned}$$

3. Use the Taylor series generated by $f(x) = x \tan^{-1} x$ to determine the value of $f^{(208)}(0)$.

Solution: We start with the Taylor series generated by $g(x) = \tan^{-1} x$. We then multiply that by x

$$\begin{aligned} \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{on } [-1, 1] \\ x \tan^{-1} x &= x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+1} \quad \text{on } [-1, 1] \end{aligned}$$

Let us now focus on the 208 degree term

$$\begin{aligned} \frac{f^{(208)}(0)}{208!} x^{208} &= (-1)^{103} \frac{x^{208}}{207} \\ \frac{f^{(208)}(0)}{208!} &= -\frac{1}{207} \\ f^{(208)}(0) &= -\frac{208!}{207} = -208 \cdot 206! \end{aligned}$$

4. Consider the Taylor series generated by $f(x) = e^x$.

a) Let i be the complex number with $i^2 = -1$. Simplify the expressions $i^2, i^3, i^4, i^5, i^6, \dots$

Solution:

$$\begin{aligned} i^3 &= (i^2) i = (-1) i = -i \\ i^4 &= (i^3) i = (-i) i = -i^2 = -(-1) = 1 \\ i^5 &= (i^4) i = (1) i = i \\ i^6 &= (i^5) i = (i) i = i^2 = -1 \end{aligned}$$

In general, if n is an integer, then

$$\begin{aligned} i^{4n} &= (i^4)^n = 1^n = 1 \\ i^{4n+1} &= i^{4n} \cdot i^1 = (i^4)^n \cdot i = 1^n \cdot i = 1 \cdot i = i \\ i^{4n+2} &= i^{4n} \cdot i^2 = (i^4)^n \cdot i^2 = 1^n (-1) = 1 (-1) = -1 \\ i^{4n+3} &= i^{4n} \cdot i^3 = (i^4)^n \cdot i^3 = 1^n \cdot i^3 = 1 (-i) = -i \end{aligned}$$

b) Write the Taylor series generated by $f(x) = e^x$ and evaluate it at $x = i\theta$ to prove that $e^{i\theta} = \cos \theta + i \sin \theta$

Solution:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} \\ e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} \end{aligned}$$

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} + \dots\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n i \frac{\theta^{2n+1}}{(2n+1)!} \\ e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \\ e^{i\theta} &= \cos \theta + i \sin \theta \end{aligned}$$

c) Use the result $e^{i\theta} = \cos \theta + i \sin \theta$ to compute $e^{i\pi}$

Solution: $e^{i\pi} = \cos \pi + i \sin \pi = -1$