

Theorem: (**The Intermediate Value Theorem**) If f is continuous on a closed interval $[a, b]$ and if $f(a) < 0$ and $f(b) > 0$, then there exists c in (a, b) so that $f(c) = 0$.

Theorem: (**The Intermediate value Theorem for Continuous Functions**) If f is continuous on a closed interval $[a, b]$ and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

Applications of the Intermediate Value Theorem

1. Prove that the function $f(x) = x^5 - 3x^4 + 8x^3 - x - 2$ has at least one zero in the interval $[0, 1]$.
2. Prove that the function $f(x) = 6x^4 + x^3 - 25x^2 - 4x + 4$ has at least two zeroes in the interval $[-1, 1]$.
3. Prove that all polynomials with an odd degree have at least one zero.

Solutions

1. f is continuous on $[0, 1]$, $f(0) = -2$, and $f(1) = 3$. By the Intermediate Value Theorem, f has a zero in $[0, 1]$.
2. f is continuous on $[-1, 1]$. $f(-1) = -12$ and $f(0) = 4$. By the Intermediate Value Theorem, f has a zero in $[-1, 0]$. Also, $f(0) = 4$ and $f(1) = -18$. By the Intermediate Value Theorem, f has a zero in $[0, 1]$.
3. Suppose that f is an odd degree polynomial with a positive leading coefficient. Then $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = \infty$. Then there exist sufficiently a large negative value of x (denoted by a) so that $f(a)$ is negative. Also, there exist sufficiently a large positive value of x (denoted by b) so that $f(b)$ is positive. Since f is a polynomial, it is continuous on $\mathbb{R}$ and thus on $[a, b]$. By the Intermediate Value Theorem, f must have a zero in the interval $[a, b]$. The proof goes similarly for functions with negative leading coefficients.

Maximum-Minimum Theorems

Definition: Let f be a function with domain D . Then f has a **relative maximum value** at a point c if $f(x) \leq f(c)$ for all x in D lying in some open interval containing c . A function f has a **relative minimum value** at a point c if $f(x) \geq f(c)$ for all x in D lying in some open interval containing c .

Theorem: (First Derivative Theorem for Local Extreme Values) If f has a relative maximum or minimum value at an interior point c of its domain, and if f' is defined at c , then $f'(c) = 0$.

Proof: Suppose that f is differentiable at c and f has a local maximum at c . Then $f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$ exists and is a two-sided limit. For a sufficiently small positive value of h , $f(c+h)$ exists and $f(c+h) \leq f(c)$ since f has a local maximum value at c . Then $f(c+h) - f(c) \leq 0$. Divide that by positive h and get that $\frac{f(c+h) - f(c)}{h} \leq 0$ and so

$$\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

Now let h be a very small negative number. Then by the same argument, $f(c+h) \leq f(c)$. Divide that by a negative h and get that $\frac{f(c+h) - f(c)}{h} \geq 0$ and so

$$\lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

For the two-sided limit $f'(c)$ to exist, we must have

$$\lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$$

Since one side is less than or equal to zero and the other is greater than or equal to zero, they both must be zero.

Definition: A **critical number** of a function f is a number c in its domain such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem: (Fermat) If f has a local maximum or minimum at c , then c is a critical number of f .

Definition: Let f be a function with domain D . Then f has an **absolute maximum value** on D at a point c if $f(x) \leq f(c)$ for all x in D and an **absolute minimum value** on D at a point c if $f(x) \geq f(c)$ for all x in D .

Theorem: (**Extreme Value Theorem**) If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Closed interval method: To find absolute extrema of a continuous function f on a closed interval $[a, b]$.

- 1) Find the values of f at the critical numbers of f in $[a, b]$.
- 2) Find the values of f at the endpoints of the interval.
- 3) The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

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