

Sample Problems

Compute each of the following integrals. Assume that a and b are positive numbers.

1. $\int \sin x \, dx$

8. $\int \csc x \, dx$

15. $\int \frac{\sec(\sqrt{x})}{\sqrt{x}} \, dx$

2. $\int \cos 5x \, dx$

9. $\int \sin^2 x \, dx$

16. $\int_0^{\pi/3} \sqrt{1 + \cos 2x} \, dx$

3. $\int \cos x \sin^4 x \, dx$

10. $\int \sin^3 x \, dx$

17. $\int_0^{\pi/2} \sqrt{1 - \cos x} \, dx$

4. $\int \csc^2 x \, dx$

11. $\int \sin^4 x \, dx$

5. $\int \tan x \, dx$

12. $\int \sin^5 x \, dx$

18. $\int \tan^3 x \, dx$

6. $\int \cot x \, dx$

13. $\int \frac{1}{a^2 + b^2 x^2} \, dx$

19. $\int \sin 7x \cos 3x \, dx$

7. $\int \sec x \, dx$

14. $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$

20. $\int \sin 10x \sin 4x \, dx$

Practice Problems

1. $\int \cos 3x \, dx$

8. $\int \cos^2(2x) \, dx$

15. $\int_0^{\pi/6} \sqrt{1 - \cos 6x} \, dx$

2. $\int \sin\left(4x - \frac{\pi}{5}\right) \, dx$

9. $\int \cos^3 x \, dx$

16. $\int \sin 2a \cos 8a \, da$

3. $\int \sec \theta \tan \theta \, d\theta$

10. $\int \cos^4 x \, dx$

17. $\int \cos b \cos 11b \, db$

4. $\int \sec^2 \theta \, d\theta$

11. $\int \cos^5 x \, dx$

18. $\int \sin 6\theta \sin 14\theta \, d\theta$

5. $\int x \tan(x^2) \, dx$

12. $\int \sin x \cos^5 x \, dx$

19. $\int \cos 11m \sin 3m \, dm$

6. $\int \cot(2x - \pi) \, dx$

13. $\int \sin^3 x \cos^5 x \, dx$

7. $\int \cos^2 x \, dx$

14. $\int \tan^2 x \, dx$

Sample Problems - Answers

- 1.) $-\cos x + C$ 2.) $\frac{1}{5} \sin 5x + C$ 3.) $\frac{1}{5} \sin^5 x + C$ 4.) $-\cot x + C$ 5.) $-\ln |\cos x| + C = \ln |\sec x| + C$
- 6.) $\ln |\sin x| + C$ 7.) $\ln |\sec x + \tan x| + C$ 8.) $-\ln |\csc x + \cot x| + C$ 9.) $\frac{1}{2}x - \frac{1}{4} \sin 2x + C$
- 10.) $\frac{1}{3} \cos^3 x - \cos x + C$ 11.) $-\frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + \frac{3}{8}x + C$ 12.) $-\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$
- 13.) $\frac{1}{ab} \tan^{-1} \left(\frac{b}{a}x \right) + C$ 14.) $\sin^{-1} \left(\frac{x}{a} \right) + C$ 15.) $2 \ln |\sec(\sqrt{x}) + \tan(\sqrt{x})| + C$ 16.) $\frac{\sqrt{6}}{2}$
- 17.) $2\sqrt{2} - 2$ 18.) $\frac{1}{2} \sec^2 x + \ln |\cos x| + C$ 19.) $-\frac{1}{20} \cos 10x - \frac{1}{8} \cos 4x + C$ 20.) $\frac{1}{12} \sin 6x - \frac{1}{28} \sin 14x + C$

Practice Problems - Answers

- 1.) $\frac{1}{3} \sin 3x + C$ 2.) $-\frac{1}{4} \cos \left(4x - \frac{\pi}{5} \right) + C$ 3.) $\sec \theta + C$ 4.) $\tan \theta + C$ 5.) $\frac{1}{2} \ln |\sec(x^2)| + C$
- 6.) $\frac{1}{2} \ln |\sin(2x - \pi)| + C$ 7.) $\frac{1}{2}x + \frac{1}{4} \sin 2x + C$ 8.) $\frac{1}{2}x + \frac{1}{8} \sin 4x + C$ 9.) $\sin x - \frac{1}{3} \sin^3 x + C$
- 10.) $\frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$ 11.) $\frac{1}{5} \sin^5 x - \frac{2}{3} \sin^3 x + \sin x + C$ 12.) $-\frac{1}{6} \cos^6 x + C$
- 13.) $-\frac{1}{6} \cos^6 x + \frac{1}{8} \cos^8 x + C$ 14.) $-x + \tan x + C$ 15.) $\frac{\sqrt{2}}{3}$ 16.) $\frac{1}{12} \cos 6a - \frac{1}{20} \cos 10a + C$
- 17.) $\frac{1}{20} \sin 10b + \frac{1}{24} \sin 12b + C$ 18.) $\frac{1}{16} \sin 8\theta - \frac{1}{40} \sin 20\theta + C$ 19.) $\frac{1}{16} \cos 8m - \frac{1}{28} \cos 14m + C$

Sample Problems - Solutions

1. $\int \sin x \, dx$

Solution: This is a basic integral we know from differentiating basic trigonometric functions. Since $\frac{d}{dx} \cos x = -\sin x$, clearly $\frac{d}{dx} (-\cos x) = \sin x$ and so $\int \sin x \, dx = \boxed{-\cos x + C}$.

2. $\int \cos 5x \, dx$

Solution: We know that $\frac{d}{dx} \cos x = -\sin x + C$. We will use substitution. Let $u = 5x$ and then $du = 5dx$ and so $\frac{du}{5} = dx$.

$$\int \cos 5x \, dx = \int \cos u \left(\frac{du}{5} \right) = \frac{1}{5} \int \cos u \, du = \boxed{\frac{1}{5} \sin 5x + C}$$

Note: Once we have enough practice, there is no need to perform this substitution in writing. We can just simply write $\int \cos 5x \, dx = \frac{1}{5} \sin 5x + C$.

3. $\int \cos x \sin^4 x \, dx$

Solution: Let $u = \sin x$. Then $du = \cos x \, dx$.

$$\int \cos x \sin^4 x \, dx = \int \sin^4 x (\cos x \, dx) = \int u^4 \, u = \frac{1}{5} u^5 + C = \boxed{\frac{1}{5} \sin^5 x + C}$$

4. $\int \csc^2 x \, dx$

Solution: We need to remember that $\frac{d}{dx} \cot x = -\csc^2 x$.

$$\int \csc^2 x \, dx = - \int -\csc^2 x \, dx = \boxed{-\cot x + C}$$

5. $\int \tan x \, dx$

Solution: Let $u = \cos x$. Then $du = -\sin x \, dx$.

$$\begin{aligned} \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = \int \frac{1}{u} (\sin x \, dx) = \int \frac{1}{u} (-du) = - \int \frac{1}{u} \, du = -\ln |u| + C = -\ln |\cos x| + C \\ &= \ln |(\cos x)^{-1}| + C = \boxed{\ln |\sec x| + C} \end{aligned}$$

6. $\int \cot x \, dx$

Solution: Let $u = \sin x$. Then $du = \cos x \, dx$.

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{1}{u} (\cos x \, dx) = \int \frac{1}{u} \, du = \ln |u| + C = \boxed{\ln |\sin x| + C}$$

7. $\int \sec x \, dx$

Solution: $\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$

From here we will use substitution. Recall that $\frac{d}{dx} \sec x = \sec x \tan x$ and $\frac{d}{dx} \tan x = \sec^2 x$. Let $u = \sec x + \tan x$. Then $du = (\sec x \tan x + \sec^2 x) \, dx$.

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx = \int \frac{1}{u} (\sec^2 x + \sec x \tan x) \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \boxed{\ln |\sec x + \tan x| + C}$$

8. $\int \csc x \, dx$

Solution: $\int \csc x \, dx = \int \csc x \cdot \frac{\csc x + \cot x}{\csc x + \cot x} \, dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx$

From here we will use substitution. Recall that $\frac{d}{dx} \csc x = -\csc x \cot x$ and $\frac{d}{dx} \cot x = -\csc^2 x$. Let $u = \csc x + \cot x$. Then $du = (-\csc^2 x - \csc x \cot x) \, dx$.

$$\begin{aligned} \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx &= \int \frac{1}{u} (\csc^2 x + \csc x \cot x) \, dx = \int \frac{1}{u} (-du) = -\int \frac{1}{u} \, du = -\ln |u| + C \\ &= \boxed{-\ln |\csc x + \cot x| + C} \end{aligned}$$

9. $\int \sin^2 x \, dx$

Solution: Recall the double angle formula for cosine, $\cos 2x = 1 - 2\sin^2 x$. We solve this for $\sin^2 x$

$$\sin^2 x = \frac{1}{2} (1 - \cos 2x)$$

$$\begin{aligned} \int \sin^2 x \, dx &= \int \frac{1}{2} (1 - \cos 2x) \, dx = \frac{1}{2} \left(\int 1 \, dx - \int \cos 2x \, dx \right) = \frac{1}{2} \left(x + C_1 - \frac{1}{2} \sin 2x + C_2 \right) \\ &= \boxed{\frac{1}{2}x - \frac{1}{4} \sin 2x + C} \end{aligned}$$

10. $\int \sin^3 x \, dx$

Solution:

$$\int \sin^3 x \, dx = \int \sin x \sin^2 x \, dx = \int \sin x (1 - \cos^2 x) \, dx$$

Let $u = \cos x$. Then $du = -\sin x \, dx$

$$\begin{aligned} \int \sin^3 x \, dx &= \int \sin x (1 - \cos^2 x) \, dx = \int (1 - \cos^2 x) (\sin x \, dx) = \int (1 - u^2) (-du) = \int (u^2 - 1) \, du \\ &= \frac{1}{3} u^3 - u + C = \boxed{\frac{1}{3} \cos^3 x - \cos x + C} \end{aligned}$$

11. $\int \sin^4 x \, dx$

Solution: We use the double angle formula for cosine to express $\sin^2 x$.

$$\cos 2x = 1 - 2\sin^2 x \implies \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx = \int \left(\frac{1}{2}(1 - \cos 2x)\right)^2 \, dx = \frac{1}{4} \int (1 - \cos 2x)^2 \, dx = \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \, dx$$

We use the double angle formula for cosine again to express $\cos^2 2x$.

$$\cos 4x = 2\cos^2 2x - 1 \implies \cos^2 2x = \frac{1}{2}(\cos 4x + 1)$$

$$\begin{aligned} \int \sin^4 x \, dx &= \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \, dx = \frac{1}{4} \int \left(1 - 2\cos 2x + \frac{1}{2}(\cos 4x + 1)\right) \, dx \\ &= \int \left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x + \frac{1}{8}\right) \, dx = \int \left(-\frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x + \frac{3}{8}\right) \, dx \\ &= -\frac{1}{2}\left(\frac{1}{2}\right)\sin 2x + \frac{1}{8}\left(\frac{1}{4}\right)\sin 4x + \frac{3}{8}x + C = \boxed{-\frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + \frac{3}{8}x + C} \end{aligned}$$

12. $\int \sin^5 x \, dx$

Solution: This method works with odd powers of $\sin x$ or $\cos x$. We will separate one factor of $\sin x$ from the rest which will be expressed in terms of $\cos x$.

$$\begin{aligned} \int \sin^5 x \, dx &= \int \sin x \sin^4 x \, dx = \int \sin x \sin^4 x \, dx = \int \sin x (\sin^2 x)^2 \, dx = \int \sin x (1 - \cos^2 x)^2 \, dx \\ &= \int \sin x (1 - 2\cos^2 x + \cos^4 x) \, dx \end{aligned}$$

We proceed with substitution. Let $u = \cos x$. Then $du = -\sin x \, dx$.

$$\begin{aligned} \int \sin^5 x \, dx &= \int \sin x (1 - 2\cos^2 x + \cos^4 x) \, dx = \int (1 - 2\cos^2 x + \cos^4 x) (\sin x \, dx) \\ &= \int (1 - 2u^2 + u^4) (-du) = \int (-1 + 2u^2 - u^4) \, du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C \\ &= \boxed{-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C} \end{aligned}$$

13. $\int \frac{1}{a^2 + b^2 x^2} \, dx$

Solution: The basic integral here is $\int \frac{1}{x^2 + 1} \, dx = \tan^{-1} x + C$. We need a substitution under which $a^2 x^2 = b^2 u^2$. This would be convenient because then

$$\frac{1}{a^2 x^2 + b^2} = \frac{1}{b^2 u^2 + b^2} = \frac{1}{b^2} \cdot \frac{1}{u^2 + 1}$$

So we will pursue this substitution. We solve $a^2 x^2 = b^2 u^2$ for a possible value of u and obtain $u = \frac{a}{b}x$. Then

$$du = \frac{a}{b}dx \text{ and so } \frac{b}{a}du = dx.$$

$$\begin{aligned} \int \frac{1}{a^2 x^2 + b^2} \, dx &= \int \frac{1}{b^2 u^2 + b^2} \left(\frac{b}{a}du\right) = \int \frac{1}{b^2} \cdot \frac{1}{u^2 + 1} \cdot \frac{b}{a} \, du = \frac{b}{ab^2} \int \frac{1}{u^2 + 1} \, du = \frac{1}{ab} \tan^{-1} u + C \\ &= \boxed{\frac{1}{ab} \tan^{-1} \left(\frac{b}{a}x\right) + C} \end{aligned}$$

14. $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

Solution: The basic integral here is $\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + C$. We need a substitution under which $x^2 = a^2 u^2$. This would be useful because then

$$\frac{1}{\sqrt{a^2 - x^2}} = \frac{1}{\sqrt{a^2 - a^2 u^2}} = \frac{1}{\sqrt{a^2 (1 - u^2)}} = \frac{1}{a\sqrt{1 - u^2}}$$

So we will pursue this substitution. We solve $x^2 = a^2 u^2$ for a possible value of u and obtain $x = au$ and $dx = a du$.

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{\sqrt{a^2 - a^2 u^2}} (a du) = \int \frac{a}{a\sqrt{1 - u^2}} du = \int \frac{1}{\sqrt{1 - u^2}} du = \sin^{-1} u + C = \boxed{\sin^{-1} \left(\frac{x}{a} \right) + C}$$

15. $\int \frac{\sec(\sqrt{x})}{\sqrt{x}} dx$

Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} dx$.

$$\begin{aligned} \int \frac{\sec(\sqrt{x})}{\sqrt{x}} dx &= 2 \int \frac{\sec(\sqrt{x})}{2\sqrt{x}} dx = 2 \int \sec(\sqrt{x}) \left(\frac{1}{2\sqrt{x}} dx \right) = 2 \int \sec u du = 2 \ln |\sec u + \tan u| + C \\ &= \boxed{2 \ln |\sec(\sqrt{x}) + \tan(\sqrt{x})| + C} \end{aligned}$$

16. $\int_0^{\pi/3} \sqrt{1 + \cos 2x} dx$

Solution: We will yet again use the double angle formula for cosine, this time to eliminate the square root.

$$\cos 2x = 2 \cos^2 x - 1 \implies 2 \cos^2 x = \cos 2x + 1$$

$$\int_0^{\pi/3} \sqrt{1 + \cos 2x} dx = \int_0^{\pi/3} \sqrt{2 \cos^2 x} dx = \sqrt{2} \int_0^{\pi/3} \sqrt{\cos^2 x} dx = \sqrt{2} \int_0^{\pi/3} |\cos x| dx$$

Since $f(x) = \cos x$ is positive on $\left[0, \frac{\pi}{3}\right]$, we can simplify $|\cos x| = \cos x$

$$\sqrt{2} \int_0^{\pi/3} |\cos x| dx = \sqrt{2} \int_0^{\pi/3} \cos x dx = \sqrt{2} \left(\sin x \Big|_0^{\pi/3} \right) = \sqrt{2} \left(\sin \frac{\pi}{3} - \sin 0 \right) = \sqrt{2} \left(\frac{\sqrt{3}}{2} - 0 \right) = \boxed{\frac{\sqrt{6}}{2}}$$

$$17. \int_0^{\pi/2} \sqrt{1 - \cos x} \, dx$$

Solution:

$$\cos 2\theta = 1 - 2\sin^2 \theta \implies 2\sin^2 \theta = 1 - \cos 2\theta$$

We substitute $\theta = \frac{x}{2}$ into this and obtain

$$2\sin^2 \frac{x}{2} = 1 - \cos x$$

$$\int_0^{\pi/2} \sqrt{1 - \cos x} \, dx = \int_0^{\pi/2} \sqrt{2\sin^2 \frac{x}{2}} \, dx = \sqrt{2} \int_0^{\pi/2} \left| \sin \frac{x}{2} \right| \, dx$$

Since $f(x) = \sin \frac{x}{2}$ is non-negative on $\left[0, \frac{\pi}{2}\right]$, we can simplify $\left| \sin \frac{x}{2} \right| = \sin \frac{x}{2}$

$$\begin{aligned} \sqrt{2} \int_0^{\pi/2} \sin \frac{x}{2} \, dx &= \sqrt{2} \left(-2 \cos \frac{x}{2} \Big|_0^{\pi/2} \right) = -2\sqrt{2} \left(\cos \frac{x}{2} \Big|_0^{\pi/2} \right) = -2\sqrt{2} \left(\cos \frac{\pi}{4} - \cos 0 \right) \\ &= -2\sqrt{2} \left(\frac{\sqrt{2}}{2} - 1 \right) = -2 + 2\sqrt{2} = \boxed{2\sqrt{2} - 2} \end{aligned}$$

$$18. \int \tan^3 x \, dx$$

Solution: Let $u = \cos x$. Then $du = -\sin x \, dx$

$$\begin{aligned} \int \tan^3 x \, dx &= \int \frac{\sin^3 x}{\cos^3 x} \, dx = \int \sin x \frac{\sin^2 x}{\cos^3 x} \, dx = \int \frac{1 - \cos^2 x}{\cos^3 x} \sin x \, dx = \int \frac{1 - u^2}{u^3} (-du) = \int \frac{u^2 - 1}{u^3} \, du \\ &= \int \frac{u^2}{u^3} - \frac{1}{u^3} \, du = \int \frac{1}{u} - u^{-3} \, du = \ln |u| - \frac{u^{-2}}{-2} + C = \ln |u| + \frac{1}{2u^2} + C \\ &= \ln |\cos x| + \frac{1}{2} \sec^2 x + C \end{aligned}$$

$$19. \int \sin 7x \cos 3x \, dx$$

Solution: We will use the product-to-sum identities to transform this product into a sum. We write the sine formula for the sum and the difference of these two angles.

$$\begin{aligned} \sin 10x &= \sin(7x + 3x) = \sin 7x \cos 3x + \cos 7x \sin 3x \\ \sin 4x &= \sin(7x - 3x) = \sin 7x \cos 3x - \cos 7x \sin 3x \end{aligned}$$

We will add the two equations

$$\begin{aligned} \sin 10x + \sin 4x &= 2\sin 7x \cos 3x \\ \frac{1}{2}(\sin 10x + \sin 4x) &= \sin 7x \cos 3x \end{aligned}$$

We can very easily integrate $\frac{1}{2}(\sin 10x + \sin 4x)$

$$\begin{aligned} \int \sin 7x \cos 3x \, dx &= \int \frac{1}{2}(\sin 10x + \sin 4x) \, dx = \frac{1}{2} \int \sin 10x + \sin 4x \, dx \\ &= \frac{1}{2} \left(\frac{1}{10} \right) (-\cos 10x) + \frac{1}{2} \left(\frac{1}{4} \right) (-\cos 4x) + C = \boxed{-\frac{1}{20} \cos 10x - \frac{1}{8} \cos 4x + C} \end{aligned}$$

$$20. \int \sin 10x \sin 4x \, dx$$

Solution: We will use the product-to-sum identities to transform this product into a sum. We write the cosine formula for the sum and the difference of these two angles.

$$\begin{aligned}\cos 14x &= \cos(10x + 4x) = \cos 10x \cos 4x - \sin 10x \sin 4x \\ \cos 6x &= \cos(10x - 4x) = \cos 10x \cos 4x + \sin 10x \sin 4x\end{aligned}$$

We will subtract the first equation from the first one

$$\begin{aligned}\cos 6x - \cos 14x &= 2 \sin 10x \sin 4x \\ \frac{1}{2}(\cos 6x - \cos 14x) &= \sin 10x \sin 4x\end{aligned}$$

We can very easily integrate $\frac{1}{2}(\cos 6x - \cos 14x)$

$$\begin{aligned}\int \sin 10x \sin 4x \, dx &= \int \frac{1}{2}(\cos 6x - \cos 14x) \, dx = \frac{1}{2} \int \cos 6x - \cos 14x \, dx \\ &= \frac{1}{2} \left(\frac{1}{6} \right) (\sin 6x) - \frac{1}{2} \left(\frac{1}{14} (\sin 14x) \right) + C = \boxed{\frac{1}{12} \sin 6x - \frac{1}{28} \sin 14x + C}\end{aligned}$$