Sample Problems

Compute each of the following limits.

1. $\displaystyle\lim_{x\to 2}\left(3x^2-5x+2\right)$

2. $\displaystyle\lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}$

3. $\displaystyle\lim_{x\to 6}\frac{\frac{1}{x}-\frac{1}{6}}{x-6}$

4. $\displaystyle\lim_{x\to 7}\left(2x-|x-7|\right)$

5. $\displaystyle\lim_{x\to 2^-}\frac{1}{x^2-9}$

6. $\displaystyle\lim_{x\to -3^-}\frac{1}{x^2-9}$

7. $\displaystyle\lim_{x\to -3^+}\frac{1}{x^2-9}$

8. $\displaystyle\lim_{x\to -3}\frac{1}{x^2-9}$

9. $\displaystyle\lim_{x\to 5^-}\frac{x^2-4x-5}{x^2-25}$

10. $\displaystyle\lim_{x\to 5^+}\frac{x^2-4x-5}{x^2-25}$

11. $\displaystyle\lim_{x\to 5}\frac{x^2-4x-5}{x^2-25}$

12. $\displaystyle\lim_{x\to -5}\frac{x^2-4x-5}{x^2-25}$

13. $\displaystyle\lim_{x\to 1}2\sqrt{x-1}$

14. $\displaystyle\lim_{x\to 0}\frac{x+2}{x^3-5x^2}$

15. $\displaystyle\lim_{x\to 0}\frac{x}{5x^3-x^4}$

Practice Problems

Compute each of the following limits.

1. $\displaystyle\lim_{x\to 8^-}\frac{1}{\sqrt{x+1}-3}$

2. $\displaystyle\lim_{x\to 8^+}\frac{1}{\sqrt{x+1}-3}$

3. $\displaystyle\lim_{x\to 8}\frac{1}{\sqrt{x+1}-3}$

4. $\displaystyle\lim_{x\to 2^+}\frac{x^2+3x}{9-x^2}$

5. $\displaystyle\lim_{x\to 3^-}\frac{x^2+3x}{9-x^2}$

6. $\displaystyle\lim_{x\to 3^+}\frac{x^2+3x}{9-x^2}$

7. $\displaystyle\lim_{x\to 3}\frac{x^2+3x}{9-x^2}$

8. $\displaystyle\lim_{x\to -3^-}\frac{x^2+3x}{9-x^2}$

9. $\displaystyle\lim_{x\to -3^+}\frac{x^2+3x}{9-x^2}$

10. $\displaystyle\lim_{x\to -3^+}\frac{x^2+3x}{9-x^2}$

11. $\displaystyle\lim_{x\to 4^+}\frac{\frac{1}{x}-\frac{1}{4}}{x^2-16}$

12. $\displaystyle\lim_{x\to 4^-}\frac{\sqrt{x}-2}{x-4}$

13. $\displaystyle\lim_{x\to 3^+}\frac{\sqrt{x+1}-2}{3-x}$

14. $\displaystyle\lim_{x\to 3^-}\frac{\sqrt{x+1}-2}{3-x}$

15. $\displaystyle\lim_{x\to 15^+}\frac{1}{\sqrt{x-15}}$

16. $\displaystyle\lim_{x\to 15^-}\frac{1}{\sqrt{x-15}}$

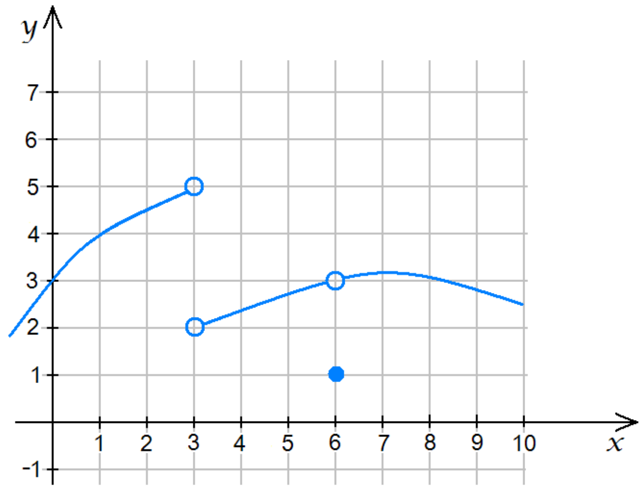
17. Determine the limits of the function based on its graph shown.

a) $\lim_{x \to 3^-} f(x)$ e) $\lim_{x \to 6^-} f(x)$

b) $\lim_{x \to 3^+} f(x)$ f) $\lim_{x \to 6^+} f(x)$

c) $\lim_{x \to 3} f(x)$ g) $\lim_{x \to 6} f(x)$

d) $f(3)$ h) $f(6)$



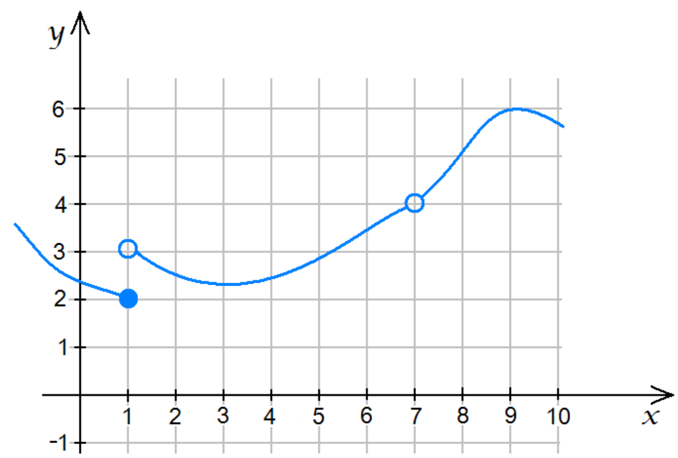
18. Determine each of the following based on the picture shown.

a) $\lim_{x \to 1^-} f(x)$ e) $\lim_{x \to 7^-} f(x)$

b) $\lim_{x \to 1^+} f(x)$ f) $\lim_{x \to 7^+} f(x)$

c) $\lim_{x \to 1} f(x)$ g) $\lim_{x \to 7} f(x)$

d) $f(1)$ h) $f(7)$



19. Determine each of the following based on the picture shown.

a) $\lim_{x \to -5^-} f(x)$ i) $\lim_{x \to 1^-} f(x)$

b) $\lim_{x \to -5^+} f(x)$ j) $\lim_{x \to 1^+} f(x)$

c) $\lim_{x \to -5} f(x)$ k) $\lim_{x \to 1} f(x)$

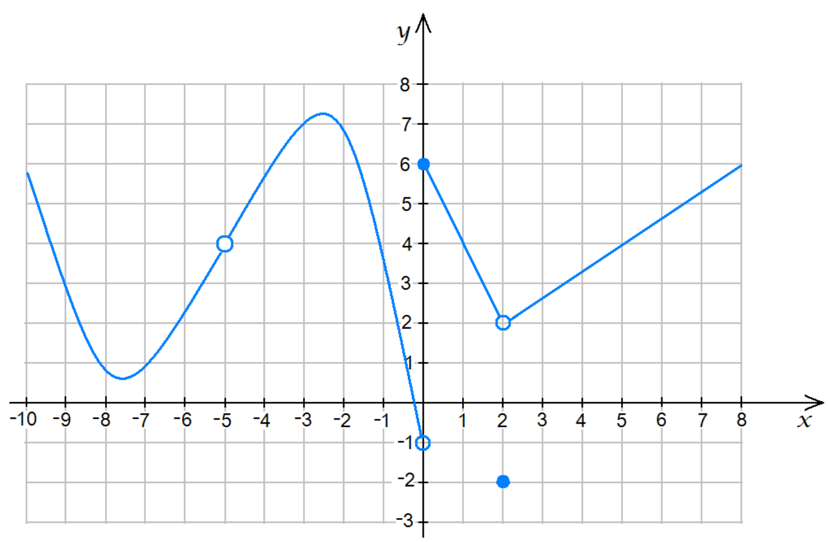
d) $f(-5)$ l) $f(1)$

e) $\lim_{x \to 0^-} f(x)$ m) $\lim_{x \to 2^-} f(x)$

f) $\lim_{x \to 0^+} f(x)$ n) $\lim_{x \to 2^+} f(x)$

g) $\lim_{x \to 0} f(x)$ o) $\lim_{x \to 2} f(x)$

h) $f(0)$ p) $f(2)$



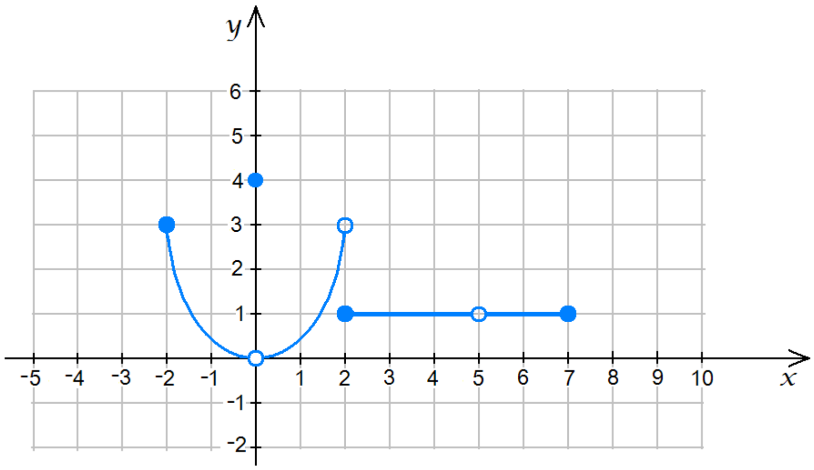
20. Determine each of the following based on the picture shown.

a) $\lim_{x\to -2^-} f(x)$ e) $\lim_{x\to 0^-} f(x)$ i) $\lim_{x\to 2^-} f(x)$ m) $\lim_{x\to 5^-} f(x)$ q) $\lim_{x\to 7^-} f(x)$

b) $\lim_{x\to -2^+} f(x)$ f) $\lim_{x\to 0^+} f(x)$ j) $\lim_{x\to 2^+} f(x)$ n) $\lim_{x\to 5^+} f(x)$ r) $\lim_{x\to 7^+} f(x)$

c) $\lim_{x\to -2} f(x)$ g) $\lim_{x\to 0} f(x)$ k) $\lim_{x\to 2} f(x)$ o) $\lim_{x\to 5} f(x)$ s) $\lim_{x\to 7} f(x)$

d) $f(-2)$ h) $f(0)$ l) $f(2)$ p) $f(5)$ t) $f(7)$



Sample Problems - Answers

1.) 4 2.) $\frac{1}{6}$ 3.) $-\frac{1}{36}$ 4.) 14 5.) $-\frac{1}{5}$ 6.) ∞ 7.) $-\infty$ 8.) undefined

9.) $\frac{3}{5}$ 10.) $\frac{3}{5}$ 11.) $\frac{3}{5}$ 12.) undefined 13.) undefined 14.) $-\infty$ 15.) ∞

Practice Problems - Answers

1.) $-\infty$ 2.) ∞ 3.) undefined 4.) 2 5.) ∞ 6.) $-\infty$ 7.) undefined 8.) $-\frac{1}{2}$

9.) $-\frac{1}{2}$ 10.) $-\frac{1}{2}$ 11.) $-\frac{1}{128}$ 12.) $\frac{1}{4}$ 13.) $-\frac{1}{4}$ 14.) $-\frac{1}{4}$ 15.) ∞ 16.) undefined

17.) a) 5 b) 2 c) undefined d) undefined e) 3 f) 3 g) 3 h) 1

18.) a) 2 b) 3 c) undefined d) 2 e) 4 f) 4 g) 4 h) undefined

19.) a) 4 b) 4 c) 4 d) undefined e) -1 f) 6 g) undefined h) 6 i) 4 j) 4 k) 4 l) 4
m) 2 n) 2 o) 2 p) -2

20.) a) undefined b) 3 c) undefined d) 3 e) 0 f) 0 g) 0 h) 4 i) 3 j) 1 k) undefined
l) 1 m) 1 n) 1 o) 1 p) undefined q) 1 r) undefined s) undefined t) 1

Sample Problems - Solutions

1.) $\lim_{x\to 2}\left(3x^2-5x+2\right)$

Solution: Let us first substitute $x=2$ into the expression.

$$3\cdot 2^2-5\cdot 2+2=12-10+2=4$$

Since the result is a well-defined number, we have a two-sided limit:

$$\begin{aligned}\lim_{x\to 2^-}\left(3x^2-5x+2\right)&=4 \quad \text{is the left-hand side limit, and}\\ \lim_{x\to 2^+}\left(3x^2-5x+2\right)&=4 \quad \text{is the right-hand side limit, and thus}\\ \lim_{x\to 2}\left(3x^2-5x+2\right)&=4 \quad \text{is the two-sided limit.}\end{aligned}$$

2.) $\lim_{x\to 5}\dfrac{\sqrt{x+4}-3}{x-5}$

Solution: Let us first substitute $x=5$ into the expression.

$$\frac{\sqrt{5+4}-3}{5-5}=\frac{3-3}{0}=\frac{0}{0} \quad \text{undefined}$$

This is not the end of the problem. Our result does not indicate that the limit doesn't exist. This expression is an **indeterminate**. In case of an indeterminate, we have to manipulate the expression until it is in a form so that we can evaluate the limit. The methods of manipulation depends on the given problem. In this case, we will use the conjugate of the numerator. We will multiply both numerator and denominator by $\sqrt{x+4}+3$.

$$\begin{aligned}\lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}&=\lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}\cdot\frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}=\lim_{x\to 5}\frac{\left(\sqrt{x+4}-3\right)\left(\sqrt{x+4}+3\right)}{(x-5)\left(\sqrt{x+4}+3\right)}\\ &=\lim_{x\to 5}\frac{\left(\sqrt{x+4}\right)^2-3^2}{(x-5)\left(\sqrt{x+4}+3\right)}=\lim_{x\to 5}\frac{x+4-9}{(x-5)\left(\sqrt{x+4}+3\right)}=\lim_{x\to 5}\frac{x-5}{(x-5)\left(\sqrt{x+4}+3\right)}\\ &=\lim_{x\to 5}\frac{1}{\sqrt{x+4}+3}\end{aligned}$$

Although we did not change the value of this expression (after all we only multiplied it by 1), this is no longer an indeterminate. We substitute $x=5$ into this new expression:

$$\frac{1}{\sqrt{5+4}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

Since the result is a well-defined number, we have a two-sided limit:
$\lim_{x\to 5^-}\dfrac{1}{\sqrt{x+4}+3}=\dfrac{1}{\sqrt{5+4}+3}=\dfrac{1}{\sqrt{9}+3}=\dfrac{1}{6}$ is the left-hand side limit, $\lim_{x\to 5^+}\dfrac{1}{\sqrt{x+4}+3}=\dfrac{1}{6}$ is the right-hand side limit, and thus $\lim_{x\to 5}\dfrac{1}{\sqrt{x+4}+3}=\dfrac{1}{6}$ is the two-sided limit. Thus $\lim_{x\to 5}\dfrac{\sqrt{x+4}-3}{x-5}=\dfrac{1}{6}$.

3.) $\lim\limits_{x\to 6}\dfrac{\dfrac{1}{x}-\dfrac{1}{6}}{x-6}$

Solution: Let us first substitute $x=6$ into the expression.

$$\frac{\dfrac{1}{6}-\dfrac{1}{6}}{6-6}=\frac{0}{0}=\text{undefined}$$

This expression is an **indeterminate**. We only simplify this complex fraction.

$$\lim_{x\to 6}\frac{\dfrac{1}{x}-\dfrac{1}{6}}{x-6}=\lim_{x\to 6}\frac{\dfrac{6}{6x}-\dfrac{x}{6x}}{x-6}=\lim_{x\to 6}\frac{\dfrac{6-x}{6x}}{x-6}=\lim_{x\to 6}\frac{6-x}{6x}\cdot\frac{1}{x-6}=\lim_{x\to 6}\frac{-(x-6)}{6x}\cdot\frac{1}{x-6}=\lim_{x\to 6}\frac{-1}{6x}$$

We substitute $x=6$ into this new expression:

$$\lim_{x\to 6}\frac{-1}{6x}=\frac{-1}{6\cdot 6}=-\frac{1}{36}$$

Thus we have a two-sided limit:

$$\lim_{x\to 6^-}\frac{\dfrac{1}{x}-\dfrac{1}{6}}{x-6}=-\frac{1}{36},\qquad \lim_{x\to 6^+}\frac{\dfrac{1}{x}-\dfrac{1}{6}}{x-6}=-\frac{1}{36},\quad \text{and so}\quad \lim_{x\to 6}\frac{\dfrac{1}{x}-\dfrac{1}{6}}{x-6}=-\frac{1}{36}$$

4.) $\lim\limits_{x\to 7}(2x-|x-7|)$

Solution: Recall the definition of $f(x)=|x|$ first.

$$|x|=\begin{cases} x & \text{if} \quad x\ge 0\\ -x & \text{if} \quad x<0\end{cases}$$

We will compute the left limit, $\lim\limits_{x\to 7^-}(2x-|x-7|)$ first. As x approaches 7 from the left, x is less than 7. Thus $x-7$ is negative and so $|x-7|=-(x-7)$.

$$\lim_{x\to 7^-}(2x-|x-7|)=\lim_{x\to 7^-}(2x-[-(x-7)])=\lim_{x\to 7^-}(2x+x-7)=\lim_{x\to 7^-}(3x-7)=3\cdot 7-7=14$$

We will now compute the right limit, $\lim\limits_{x\to 7^+}(2x-|x-7|)$. As x approaches 7 from the right, x is greater than 7. Thus $x-7$ is positive and so $|x-7|=x-7$.

$$\lim_{x\to 7^+}(2x-|x-7|)=\lim_{x\to 7^+}(2x-(x-7))=\lim_{x\to 7^+}(2x-x+7)=\lim_{x\to 7^+}(x+7)=7+7=14$$

Because the left-hand side limit equals to the right-hand side limit, there is a two-sided limit and it is 14.

$$\lim_{x\to 7^-}(2x-|x-7|)=\lim_{x\to 7^+}(2x-|x-7|)=14\quad\Longrightarrow\quad\lim_{x\to 7}(2x-|x-7|)=14$$

5.) $\lim\limits_{x\to 2^-}\dfrac{1}{x^2-9}$

Solution: Let us first substitute $x=2$ into the expression.

$$\frac{1}{2^2-9}=\frac{1}{4-9}=\frac{1}{-5}=-\frac{1}{5}$$

Thus $\lim\limits_{x\to 2^-}\dfrac{1}{x^2-9}=-\dfrac{1}{5}$. (although the problem did not ask us to find them, our computation shows that the right-hand side limit is also $-\dfrac{1}{5}$, and thus the two-sided limit also exists and is $-\dfrac{1}{5}$.)

6.) $\displaystyle\lim_{x\to -3^-}\frac{1}{x^2-9}$

Solution: Let us first substitute $x=3$ into the expression.

$$\frac{1}{3^2-9}=\frac{1}{9-9}=\frac{1}{0}\quad \text{undefined}$$

This is not a case of an indeterminate. If we divide 1 by a very small number, the result is a very large number. Thus, the answer is either $-\infty$ or ∞. The ambiguity results from the fact that if a number is very close to zero, it may be a small negative or a small positive number. We just have to find out which one it is. $x\to -3^-$ means that x is very close to -3, and that x is less than -3. The factor $x-3$ is then close to -6, so it is negative. The factor $x+3$ is close to zero. Since x is less than -3, $x+3$ is less than 0, thus negative.

$$\begin{aligned} x &< -3 \quad \text{add } 3 \\ x+3 &< 0 \quad \Longrightarrow \quad x+3 \text{ is negative} \end{aligned}$$

As x approaches -3 from the left, $x^2-9=(x+3)(x-3)$ is positive since both factors are negative. Thus the limit is ∞.

7.) $\displaystyle\lim_{x\to -3^+}\frac{1}{x^2-9}$

Solution: When we substitute $x=3$ into the expression (see previous problem) we get $\dfrac{1}{0}$. This means that the answer is either $-\infty$ or ∞; and we only have to figure out which one. Keep in mind that $x\to -3^+$ means that x is very close to -3, and that x is greater than -3.

$$x^2-9=(x+3)(x-3)$$

$x>-3$ add 3	$x>-3$ subtract 3
$x+3>0$	$x-3>-6$
the factor $x+3$ is positive	the factor $x-3$ is negative. Although $x-3$ is greater than -6, but is also close to -6 and so must be negative.

As x approaches -3 from the right, $x^2-9=(x+3)(x-3)$ is negative since one factor is positive and the other is negative. Thus the limit is $-\infty$.

8.) $\displaystyle\lim_{x\to -3}\frac{1}{x^2-9}$

Solution: We have worked out the one-sided limits in the previous problems. Our results were

$$\lim_{x\to -3^-}\frac{1}{x^2-9}=\infty \quad \text{and} \quad \lim_{x\to -3^+}\frac{1}{x^2-9}=-\infty$$

Since these limits are not equal, the two-sided limit does not exist. $\displaystyle\lim_{x\to -3}\frac{1}{x^2-9}=\text{undefined}$

9.) $\displaystyle\lim_{x\to 5^-}\frac{x^2-4x-5}{x^2-25}$

Solution: Let us first substitute $x=5$ into the expression.

$$\frac{5^2-4\cdot 5-5}{5^2-25}=\frac{25-20-5}{25-25}=\frac{0}{0}\quad \text{undefined}$$

This expression is an **indeterminate**. In case of a $\dfrac{0}{0}$ indeterminate in a rational function, we must factor both numerator and denominator and cancel common factors. If a polynomial takes a zero at $x=5$, it is then divisible by $x-5$. Thus both numerator and denominator are divisible by $x-5$. Once we cancelled all common factors, the expression will no longer be an indeterminate.

$$\lim_{x\to 5^-}\frac{x^2-4x-5}{x^2-25}=\lim_{x\to 5^-}\frac{(x-5)(x+1)}{(x-5)(x+5)}=\lim_{x\to 5^-}\frac{x+1}{x+5}$$

We can now substitute $x=5$ into this new expression

$$\frac{5+1}{5+5}=\frac{6}{10}=\frac{3}{5}$$

10.) $\lim_{x\to 5^+} \dfrac{x^2-4x-5}{x^2-25}$

Solution: We actually have the answer to this question. In the previous problem, the expression was first an indetermiate. After cancellation, we were able to substitute $x=5$ and that indicates a two-sided limit. Thus $\lim_{x\to 5^+} \dfrac{x^2-4x-5}{x^2-25} = \dfrac{3}{5}$.

11.) $\lim_{x\to 5} \dfrac{x^2-4x-5}{x^2-25}$

Solution: Since the left-hand side limit and right-hand side limit (worked out in the previous problems) both exist and they are equal, we have a two-sided limit

$$\lim_{x\to 5^-} \frac{x^2-4x-5}{x^2-25} = \frac{3}{5} \quad \text{and} \quad \lim_{x\to 5^+} \frac{x^2-4x-5}{x^2-25} = \frac{3}{5} \quad \Longrightarrow \quad \lim_{x\to 5} \frac{x^2-4x-5}{x^2-25} = \frac{3}{5}$$

12.) $\lim_{x\to -5} \dfrac{x^2-4x-5}{x^2-25}$

Solution: Let us first substitute $x=-5$ into the expression.

$$\frac{(-5)^2-4(-5)-5}{(-5)^2-25} = \frac{25+20-5}{25-25} = \frac{40}{0} \quad \text{undefined}$$

This is not a case of an indeterminate. If we divide 1 by a very small number, the result is a very large number. Thus, the answer is either $-\infty$ or ∞ on either sides of -5. We have to compute the left-hand and right-hand limits separately. Let us first simplify this expression:

$$\lim_{x\to -5} \frac{x^2-4x-5}{x^2-25} = \lim_{x\to -5} \frac{(x-5)(x+1)}{(x-5)(x+5)} = \lim_{x\to -5} \frac{x+1}{x+5}$$

Let us now compute the left-hand side limit. $x\to -5^-$ means that x is very close to -5, and that x is less than -5.

$x<-5$ add 5	$x<-5$ add 1
$x+5<0$	$x+1<-4$
the factor $x+5$ is negative	the factor $x+1$ is negative

As x approaches -5 from the left, $\dfrac{x+1}{x+5}$ is positive since both numerator denominator are negative. Thus the left-hand side limit is ∞, i.e. $\lim_{x\to -5^-} \dfrac{x^2-4x-5}{x^2-25} = \infty$.

Now for the right-hand limit: $x\to -5^+$ means that x is very close to -5, and that x is greater than -5.

$x>-5$ add 5	$x>-5$ add 1
$x+5>0$	$x+1>-4$
the factor $x+5$ is positive	the factor $x+1$ is negative. Although $x+1$ is greater than -4, but is also close to -4 and so must be negative.

As x approaches -5 from the right, $\dfrac{x+1}{x+5}$ is negative since the numerator is negative and the denominator is positive. Thus the right-hand side limit is $-\infty$, i.e. $\lim_{x\to -5^+} \dfrac{x^2-4x-5}{x^2-25} = -\infty$. Because of the two one-sided limits are not equal, the two-sided limit does not exist. $\lim_{x\to -5} \dfrac{x^2-4x-5}{x^2-25} = \text{undefined}$.

13.) $\lim_{x\to 1} 2\sqrt{x-1}$

Solution: When we substitute $x=1$, the result is zero. However, we do not have a two-sided limit. As x approaches 1 from the left, $\sqrt{x-1}$ is undefined since $x-1$ is negative. Thus

$$\lim_{x\to 1^-} 2\sqrt{x-1} = \text{undefined} \quad \text{and} \quad \lim_{x\to 1^+} 2\sqrt{x-1} = 0$$

thus $\lim_{x\to 1} 2\sqrt{x-1} = \text{undefined}$.

14.) $\lim_{x\to 0} \dfrac{x+2}{x^3-5x^2}$

Solution: Let us first substitute $x=0$ into the expression.

$$\frac{0+2}{0^3-5\cdot 0} = \frac{2}{0} \quad \text{undefined}$$

The one-sided limits are ∞ or $-\infty$. Let us bring the expression first in a more convenient form, where both numerator and denominator are factored. Also, if there is cancellation, that makes the problem easier.

$$\lim_{x\to 0} \frac{x+2}{x^3-5x^2} = \lim_{x\to 0} \frac{x+2}{x^2(x-5)}$$

We compute first the left-hand side limit. As x approaches 0 from the left, it is very close to zero, and it is also less than zero. Then $x+2$ is positive, $x-5$ is negative, and x^2 is positive. Thus $\lim_{x\to 0^-} \dfrac{x+2}{x^2(x-5)} = -\infty$. As x approaches 0 from the right, it is very close to zero, and it is also greater than zero. Then $x+2$ is positive, $x-5$ is negative, and x^2 is positive. Thus $\lim_{x\to 0^+} \dfrac{x+2}{x^2(x-5)} = -\infty$. Thus the two-sided limit exists and $\lim_{x\to 0} \dfrac{x+2}{x^2(x-5)} = -\infty$

15.) $\lim_{x\to 0} \dfrac{x}{5x^3-x^4}$

Solution: Let us first substitute $x=0$ into the expression. $\dfrac{0}{5\cdot 0^3-0^4} = \dfrac{0}{0}$ undefined

This expression is an **indeterminate**. In case of a $\dfrac{0}{0}$ indeterminate in a rational function, we must factor both numerator and denominator and cancel common factors. Both numerator and denominator are clearly divisible by x. Once we cancelled all common factors, the expression will no longer be an indeterminate.

$$\lim_{x\to 0} \frac{x}{5x^3-x^4} = \lim_{x\to 0} \frac{x}{-x^4+5x^3} = \lim_{x\to 0} \frac{x}{-x^3(x+5)} = \lim_{x\to 0} \frac{-1}{x^2(5x-1)}$$

We substitute $x=0$ into this new expression: $\dfrac{-1}{0^2(5\cdot 0-1)} = \dfrac{-1}{0}$ undefined

This is no longer an indeterminate, the one-sided limits are ∞ or $-\infty$. We separately compute these limits. First, the left-hand side limit: If x is less than zero and close to zero, then -1 is negative, x^2 is positive, and $5x-1$ is negative. Thus $\dfrac{-1}{x^2(5x-1)}$ is positive, and so $\lim_{x\to 0^-} \dfrac{x}{5x^3-x^4} = \infty$. For the right-hand side limit: If x is greater than zero and close to zero, then -1 is negative, x^2 is positive, and $5x-1$ is negative. Thus $\dfrac{-1}{x^2(5x-1)}$ is positive, and so $\lim_{x\to 0^+} \dfrac{x}{5x^3-x^4} = \infty$. Then the two-sided limit is also ∞.