

Sample Problems

- Compute the volume of the object we obtain by rotating the region bounded by $y = \sqrt{x}$ and $y = \frac{x}{2}$ about the x -axis.
 - Compute the volume of the object we obtain if we revolve the region about the y -axis.
- The region bounded by $y = x^2 + 1$ and $y = x + 3$ is rotated about the x -axis. Find the volume of this object.
- Let R be the region bounded by $y = x$, $y = 2x$, and $y = 6$. Find the volume of the object we obtain by revolving R about
 - the y -axis
 - the x -axis

Practice Problems

- Let R be the region bounded by $y = x$ and $y = 2\sqrt{x}$. Find the volume of the object we obtain by revolving R about
 - the x -axis
 - the y -axis
- Let R be the region in the first quadrant bounded by $x = y^3$ and $x = 4y$. Which is greater, the volume of the solid generated when R is revolved about the x -axis or the y -axis?
 Let R be the region bounded by $y = x$, $y = x + 2$, $x = 0$, and $x = 4$. Find the volume of the object we obtain by revolving R about
 - the x -axis
 - the y -axis
- The region bounded by $y = \cos x$ and $y = \sin x$ (between $x = 0$ and $x = \frac{\pi}{4}$) is rotated about the x -axis. Find the volume of this object.
- Let R be the region bounded by $y = 0$, $y = \sqrt{\ln x}$, $y = 2$, and $x = 0$. Find the volume of the object we obtain by revolving R about the x -axis.

Answers - Sample Problems

- 1.) a) $\frac{8}{3}\pi$ b) $\frac{64}{15}\pi$ 2.) $\frac{117}{5}\pi$ 3.) a) 54π b) 72π

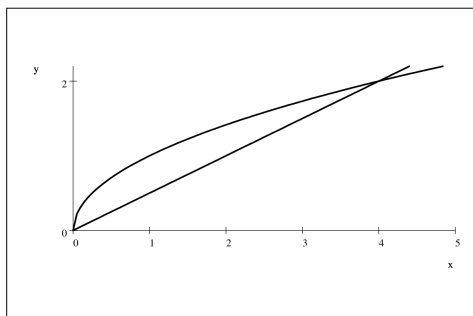
Answers - Practice Problems

- 1.) a) $\frac{32}{3}\pi$ b) $\frac{128}{15}\pi$
- 2.) about the x -axis: $V = \frac{128}{15}\pi \approx 26.808257$ about the y -axis: $V = \frac{512}{21}\pi \approx 76.595021$
- 3.) a) 48π b*) 32π 4.) $\frac{\pi}{2}$ 5.) $\pi(e^4 - 1)$

Sample Problems - Solutions

1. a) Compute the volume of the object we obtain by rotating the region bounded by $y = \sqrt{x}$ and $y = \frac{x}{2}$ about the x -axis.

Solution: We first solve for the intersection of the two graphs and obtain the intersection points $(0, 0)$ and $(4, 2)$. We are now ready to sketch the two graphs together.



The volume can be computed as

$$\begin{aligned} V &= \int_0^4 \pi \left((\sqrt{x})^2 - \left(\frac{x}{2}\right)^2 \right) dx = \pi \int_0^4 \left(x - \frac{x^2}{4} \right) dx = \pi \left(\left(\frac{x^2}{2} - \frac{x^3}{12} \right) \Big|_0^4 \right) \\ &= \pi \left(\left(\frac{4^2}{2} - \frac{4^3}{12} \right) - \left(\frac{0^2}{2} - \frac{0^3}{12} \right) \right) = \pi \left(\left(8 - \frac{64}{12} \right) - 0 \right) = \pi \left(8 - \frac{16}{3} \right) = \boxed{\frac{8}{3}\pi} \end{aligned}$$

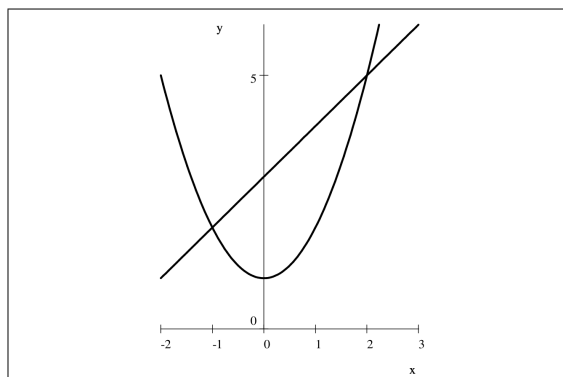
- b) Compute the volume of the object we obtain if we revolve the region about the y -axis.

Solution: Now the volume can be computed as $\int \pi (x_1^2 - x_2^2) dy$ where we express x_1 and x_2 in terms of y . The limits of the integration are the range where the horizontal slices are located. In this case, between $y = 0$ and $y = 2$. We will also need to solve $y = \sqrt{x}$ and $y = \frac{x}{2}$ for x . Clearly, if $y = \sqrt{x}$, then $x = y^2$ and if $y = \frac{x}{2}$, then $x = 2y$. Also notice that now the graph of $y = \frac{x}{2}$ is farther from the center of rotation. So the integral for the volume is

$$\begin{aligned} V &= \int_0^2 \pi (x_1^2 - x_2^2) dy = \pi \int_0^2 (2y)^2 - (y^2)^2 dy = \pi \int_0^2 4y^2 - y^4 dy = \pi \left(\frac{4y^3}{3} - \frac{y^5}{5} \right) \Big|_0^2 \\ &= \pi \left(\left(\frac{4 \cdot 2^3}{3} - \frac{2^5}{5} \right) - \left(\frac{4 \cdot 0^3}{3} - \frac{0^5}{5} \right) \right) = \pi \left(\left(\frac{32}{3} - \frac{32}{5} \right) - 0 \right) = 32\pi \left(\frac{1}{3} - \frac{1}{5} \right) \\ &= 32\pi \left(\frac{2}{15} \right) = \boxed{\frac{64}{15}\pi} \end{aligned}$$

2. The region bounded by $y = x^2 + 1$ and $y = x + 3$ is rotated about the x -axis. Find the volume of this object.

Solution: We first solve for the intersection points and obtain the points $(-1, 2)$ and $(2, 5)$.



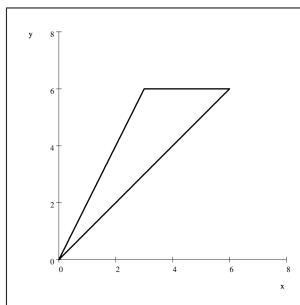
The volume can be computed as

$$\begin{aligned}
 V &= \int_{-1}^2 \pi (y_2^2 - y_1^2) dx = \int_{-1}^2 \pi ((x+3)^2 - (x^2+1)^2) dx = \pi \int_{-1}^2 (x^2 + 6x + 9 - (x^4 + 2x^2 + 1)) dx \\
 &= \pi \int_{-1}^2 (x^2 + 6x + 9 - x^4 - 2x^2 - 1) dx = \pi \int_{-1}^2 (-x^4 - x^2 + 6x + 8) dx = \pi \left(-\frac{x^5}{5} - \frac{x^3}{3} + 3x^2 + 8x \right) \Big|_{-1}^2 \\
 &= \pi \left(\left(-\frac{2^5}{5} - \frac{2^3}{3} + 3 \cdot 2^2 + 8 \cdot 2 \right) - \left(-\frac{(-1)^5}{5} - \frac{(-1)^3}{3} + 3(-1)^2 + 8(-1) \right) \right) \\
 &= \pi \left(\left(-\frac{32}{5} - \frac{8}{3} + 12 + 16 \right) - \left(\frac{1}{5} + \frac{1}{3} + 3 - 8 \right) \right) = \pi \left(\left(-\frac{136}{15} + 28 \right) - \left(\frac{8}{15} - 5 \right) \right) \\
 &= \pi \left(-\frac{136}{15} + 28 - \frac{8}{15} + 5 \right) = \pi \left(-\frac{144}{15} + 33 \right) = \pi \left(33 - \frac{48}{5} \right) = \boxed{\frac{117}{5}\pi}
 \end{aligned}$$

3. Let R be the region bounded by $y = x$, $y = 2x$, and $y = 6$. Find the volume of the object we obtain by revolving R about

a) the y -axis

Solution:

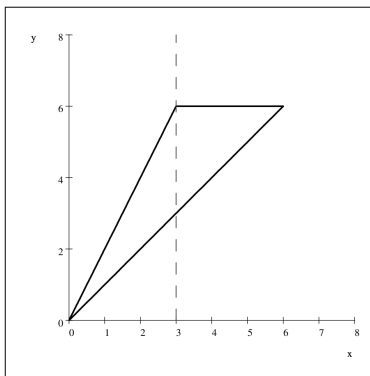


After we graphed the region, we see that $y = x$ is farther from the center of rotation. We also need to express x in terms of y for the integral. If $y = 2x$, then clearly $x = \frac{y}{2}$, and so the volume is

$$\begin{aligned}
 V &= \int_0^6 \pi (x_2^2 - x_1^2) dy = \int_0^6 \pi \left(y^2 - \left(\frac{y}{2} \right)^2 \right) dy = \pi \int_0^6 y^2 - \frac{y^2}{4} dy = \pi \int_0^6 \frac{3}{4} y^2 dy = \frac{\pi}{4} \int_0^6 3y^2 dy \\
 &= \frac{\pi}{4} \left(y^3 \Big|_0^6 \right) = \frac{\pi}{4} (6^3 - 0^3) = \frac{216}{4} \pi = \boxed{54\pi}
 \end{aligned}$$

b) the x -axis

Solution: In this case, the washer method works but we need to break down the region into two parts, as shown on the picture below. Between 0 and 3, the graph of y_2 is $y = 2x$. But between 3 and 6, the graph of y_2 is $y = 6$.



The volume is then

$$\begin{aligned}
 V &= \int \pi (y_2^2 - y_1^2) dx = \int_0^3 \pi ((2x)^2 - x^2) dx + \int_3^6 \pi (6^2 - x^2) dx = \pi \int_0^3 (4x^2 - x^2) dx + \pi \int_3^6 (-x^2 + 36) dx \\
 &= \pi \int_0^3 3x^2 dx + \pi \int_3^6 (-x^2 + 36) dx = \pi \left(\left(x^3 \Big|_0^3 \right) + \left(-\frac{x^3}{3} + 36x \Big|_3^6 \right) \right) \\
 &= \pi \left((3^3 - 0^3) + \left(-\frac{6^3}{3} + 36 \cdot 6 \right) - \left(-\frac{3^3}{3} + 36 \cdot 3 \right) \right) = \pi (27 + (-72 + 216) - (-9 + 108)) \\
 &= \pi (27 + 144 - 99) = \boxed{72\pi}
 \end{aligned}$$

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