

Sample Problems

1. We are standing on the top of a 60 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 250 N and the rope's unit weight is $2 \frac{\text{N}}{\text{m}}$.
 - a) Find a Riemann sum expressing the amount of work it takes to lift the bucket from the ground to the top of the building.
 - b) Turn the Riemann sum into an integral and evaluate it to find the work.
2. A cylindrical tank has height 10 m and base radius 3 m and it is standing on its circular base. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m^3 of water weighs 10 000 N, i.e. the density of water is $\delta = 10\,000 \frac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?
 - a) Find a Riemann sum expressing the work.
 - b) Turn the Riemann sum into an integral and evaluate it.
3. A tank, shaped like a cone has height 10 m and base radius 3 m. It is placed so that the circular part is upward. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m^3 of water weighs 10 000 N, i.e. the density of water is $\delta = 10\,000 \frac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?
 - a) Find a Riemann sum expressing the work.
 - b) Turn the Riemann sum into an integral and evaluate it.
4. The gravitational force between two objects can be computed as $F_{gr} = \frac{m_1 m_2 G}{r^2}$ where m_1 and m_2 denote the mass of each of the two objects, r is the distance between their centers of mass, and G is the universal gravitational constant, $G = 6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}$. We want to lift a satellite, measuring 800 kg, from the surface of the Earth to a height of 5000 km. How much work does this require? (Assume that Earth is a sphere with radius 6370 km and mass of $5.97 \times 10^{24} \text{ kg}$.)

Practice Problems

1. We are standing on the top of a 100 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 500 N and the rope's unit weight is $1.5 \frac{\text{N}}{\text{m}}$. Compute the work required to lift the bucket from the ground to the top of the building.
2. A cylindrical tank is of height 8 m and base radius 3 m, and it is standing on its circular base. It is full of oil, and we want to pump it all out by a pipe that is always leveled at the surface of the oil. Assume that 1 m^3 of oil weighs 9 000 N, i.e. the density of oil is $\delta = 9\,000 \frac{\text{N}}{\text{m}^3}$.
 - a) How much work does it require to pump all oil out of the tank?
 - b) How much work does it require to pump all oil out of the tank if the tank is only half full?
3. A cylindrical tank is of height H and base radius R , and it is standing on its circular base. It contains some liquid of density δ , the water level at h . We want to pump out the liquid by a pipe that is always leveled at the surface of the liquid. Compute the formula for the work required to pump all liquid out of the tank.

4. A tank, shaped like a cone has height 12 m and base radius 5 m. It is placed so that the circular part is downward. It is full of water, and we want to pump out by a pipe that is always leveled at the surface of the water. Assume that the density of water is $\delta = 10\,000 \frac{\text{N}}{\text{m}^3}$.
- How much work does it require to pump all water out of the tank?
 - How much work does it require to pump all water out of the tank if it contains water to half of its height?
5. A tank, shaped like a cone has height H and base radius R . It is placed so that the circular part is downward. It is full of water, and we want to pump out by a pipe that is always leveled at the surface of the water. Assume that the density of water is δ . Compute the formula for the work required to pump all liquid out of the tank.
6. The gravitational force between two objects can be computed as $F_{gr} = \frac{m_1 m_2 G}{r^2}$ where m_1 and m_2 denote the mass of each of the two objects, r is the distance between their centers of mass, and G is the universal gravitational constant, $G = 6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}$. We want to lift a satellite, measuring 500 kg, from the surface of the Earth to a height of 20 km. How much work does this require? (Assume that Earth is a sphere with radius 6370 km and mass of 5.97×10^{24} kg.)
7. A spherical tank of radius R is full of liquid of density δ . We want to pump out the liquid by a pipe that is always leveled at the surface of the liquid. Compute the formula for the work required to pump all liquid out of the tank.

Answers - Sample Problems

- $W = \sum_{i=0}^{n-1} (370 - 2x_i) \Delta x$
 - $\int_0^{60} (370 - 2x) dx = 18\,600 \text{ J}$
- $W = \sum_{i=0}^{n-1} (\delta \pi r^2) (10 - x_i) \Delta x$
 - $\int_0^{10} (\delta \pi r^2) (10 - x) dx = 4500\,000\pi \text{ J} \approx 1.413\,72 \times 10^7 \text{ J}$
- $W = \sum_{i=0}^{n-1} \frac{9\delta \pi x_i^2}{100} (10 - x_i) \Delta x$
 - $\int_0^{10} \frac{9\delta \pi x^2}{100} (10 - x) dx = 750\,000\pi \text{ J} \approx 2.356\,2 \times 10^6 \text{ J}$
- $2.199\,2 \times 10^{10} \text{ J}$

Answers - Practice Problems

- 57 500 J
- $2592\,000\pi \text{ J} \approx 8.143 \times 10^6 \text{ J}$
 - $1944\,000\pi \text{ J} \approx 6.107\,3 \times 10^6 \text{ J}$
- $\delta r^2 h \pi \left(H - \frac{h}{2} \right)$
- $9000\,000\pi \text{ J} \approx 2.827\,4 \times 10^7 \text{ J}$
 - $8437\,500\pi \text{ J} \approx 2.650\,7 \times 10^7 \text{ J}$
- $\frac{1}{4} \pi H^2 R^2 \delta$
- $9.782\,7 \times 10^7 \text{ J}$
- $\frac{4}{3} \pi R^4 \delta$

Sample Problems - Solutions

1. We are standing on the top of a 60 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 250 N and the rope's unit weight is $2 \frac{\text{N}}{\text{m}}$.

a) Find a Riemann sum expressing the amount of work it takes to lift the bucket from the ground to the top of the building.

Solution: Let x represent the height of the bucket. Let us partition the interval $[0, 60]$ into n equal intervals by $0 = x_0, x_1, x_2, \dots, x_n = 60$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$. We will approximate the height of the bucket to be x_i over the entire interval $[x_i, x_{i+1}]$. Then the force we have to exert while lifting the bucket from height x_i to x_{i+1} is

$$F_i = F_{\text{bucket}} + F_{\text{rope}} = 250 + 2(60 - x_i) = 370 - 2x_i$$

Then the work we have to invest when lifting the bucket from height x_i to x_{i+1} is

$$W_i = F_i \cdot s_i = (370 - 2x_i)(x_{i+1} - x_i) = (370 - 2x_i) \Delta x$$

The total work is then

$$W = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} (370 - 2x_i) \Delta x$$

b) Turn the Riemann sum into an integral and evaluate it to find the work.

Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (370 - 2x_i) \Delta x = \int_0^{60} (370 - 2x) dx = 370x - x^2 \Big|_0^{60} \\ &= (370(60) - (60)^2) - (370(0) - (0)^2) = 22\,200 - 3600 = 18\,600 \end{aligned}$$

or, the same computation with the units, quite elegant,

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \left(370 \text{ N} - 2 \frac{\text{N}}{\text{m}} x_i \right) \Delta x = \int_0^{60 \text{ m}} \left(370 \text{ N} - 2 \frac{\text{N}}{\text{m}} x \right) dx = \left(370 \text{ N} x - x^2 \frac{\text{N}}{\text{m}} \right) \Big|_{0 \text{ m}}^{60 \text{ m}} \\ &= \left(370 \text{ N} (60 \text{ m}) - (60 \text{ m})^2 \frac{\text{N}}{\text{m}} \right) - \left(370 \text{ N} (0 \text{ m}) - (0 \text{ m})^2 \frac{\text{N}}{\text{m}} \right) = 22\,200 \text{ N m} - 3600 \text{ N m} = \\ &= 18\,600 \text{ N m} = 18\,600 \text{ J} \end{aligned}$$

2. A cylindrical tank has height 10 m and base radius 3 m and it is standing on its circular base. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m^3 of water weighs 10 000 N, i.e. the density of water is $\delta = 10\,000 \frac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?

a) Find a Riemann sum expressing the work.

Solution: Let x represent the water level. Let us partition the interval $[0, 10]$ into n equal intervals by $0 = x_0, x_1, x_2, \dots, x_n = 10$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$. We will approximate the water level to be x_i over the entire interval $[x_i, x_{i+1}]$. Then the force we have to exert when pumping out the water between heights x_i and x_{i+1} is the weight of a cylindrical 'slice' with volume

$$V_i = \pi r^2 \Delta x \quad \text{the weight is then} \quad F_i = \delta V_i = \delta \pi r^2 \Delta x$$

The water needs to be lifted to the top of the tank, and so

$$s_i = 10 - x_i$$

Then the work we have to invest when lifting the slice of water from height x_i to 10 is

$$W_i = F_i \cdot s_i = (\delta \pi r^2 \Delta x) (10 - x_i) = (\delta \pi r^2) (10 - x_i) \Delta x$$

The total work is then is

$$W = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} (\delta \pi r^2) (10 - x_i) \Delta x$$

b) Turn the Riemann sum into an integral and evaluate it.

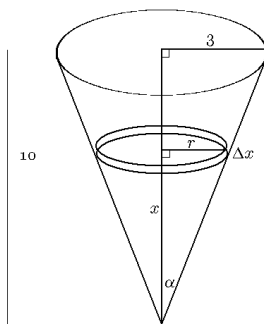
Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (\delta \pi r^2) (10 - x_i) \Delta x = \int_0^{10} (\delta \pi r^2) (10 - x) dx = \delta \pi r^2 \int_0^{10} (10 - x) dx \\ &= 10\,000\pi (9) \int_0^{10} (10 - x) dx = 90\,000\pi \left(10x - \frac{x^2}{2} \right) \Big|_0^{10} \\ &= 90\,000\pi \left[\left(10(10) - \frac{(10)^2}{2} \right) - \left(10(0) - \frac{(0)^2}{2} \right) \right] = 90\,000\pi (50) \\ &= 4500\,000\pi \approx 1.413\,72 \times 10^7 \text{ J} \end{aligned}$$

3. A tank, shaped like a cone has height 10 m and base radius 3 m. It is placed so that the circular part is upward. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m^3 of water weighs 10 000 N, i.e. the density of water is $\delta = 10\,000 \frac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?

a) Find a Riemann sum expressing the work.

Solution: Let x represent the water level. Let us partition the interval $[0, 10]$ into n equal intervals by $0 = x_0, x_1, x_2, \dots, x_n = 10$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$.



We will approximate the water level to be x_i over the entire interval $[x_i, x_{i+1}]$. Then the force we have to exert when pumping out the water between heights x_i and x_{i+1} is the weight of a cylindrical 'slice' with volume $\pi r_i^2 h$ where $h = \Delta x$ and r_i is the radius at height x_i . Clearly $\tan \alpha = \frac{r_i}{x_i} = \frac{3}{10}$ which gives us that $r_i = \frac{3}{10}x_i$

$$V_i = \pi r_i^2 \Delta x = \pi \left(\frac{3}{10} x_i \right)^2 \Delta x = \frac{9\pi x_i^2}{100} \Delta x \text{ the weight is then } F_i = \delta V_i = \frac{9\delta\pi x_i^2}{100} \Delta x$$

The water needs to be lifted to the top of the tank, and so

$$s_i = 10 - x_i$$

Then the work we have to invest when lifting the slice of water from height x_i to 10 is

$$W_i = F_i \cdot s_i = \left(\frac{9\delta\pi x_i^2}{100} \Delta x \right) (10 - x_i) = \frac{9\delta\pi x_i^2}{100} (10 - x_i) \Delta x$$

The total work is then is

$$W = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} \frac{9\delta\pi x_i^2}{100} (10 - x_i) \Delta x$$

b) Turn the Riemann sum into an integral and evaluate it.

Solution: As n approaches infinity, the Riemann sum approaches the integral

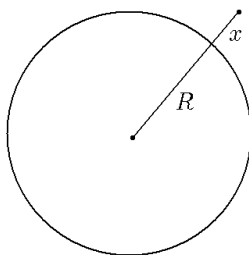
$$\begin{aligned}
 W &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \frac{9\delta\pi x_i^2}{100} (10 - x_i) \Delta x = \int_0^{10} \frac{9\delta\pi x^2}{100} (10 - x) dx = \frac{9\delta\pi}{100} \int_0^{10} x^2 (10 - x) dx \\
 &= \frac{9(10\,000)\pi}{100} \int_0^{10} x^2 (10 - x) dx = 900\pi \int_0^{10} (10x^2 - x^3) dx = 900\pi \left(\frac{10x^3}{3} - \frac{x^4}{4} \right) \Big|_0^{10} \\
 &= 900\pi \left[\left(\frac{10(10)^3}{3} - \frac{10^4}{4} \right) - \left(\frac{0^3}{3} - \frac{0^4}{4} \right) \right] = 900\pi \left(\frac{10^4}{3} - \frac{10^4}{4} \right) = 900\pi \left(\frac{1}{12} \right) 10^4 \\
 &= 750\,000\pi
 \end{aligned}$$

4. The gravitational force between two objects can be computed as $F_{gr} = \frac{m_1 m_2 G}{r^2}$ where m_1 and m_2 denote the mass of each of the two objects, r is the distance between their centers of mass, and G is the universal gravitational constant, $G = 6.67 \times 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2}$. We wanted to lift a satellite, measuring 800 kg, from the surface of the Earth to a height of 5000 km. How much work does this require? (Assume that Earth is a sphere with radius 6370 km and mass of 5.97×10^{24} kg.)

Solution: Let us denote the radius of Earth by R , the mass of the satellite by m and that of the Earth by M . Suppose that the satellite is at a distance x from the surface, and we compute the work it requires to move the satellite just a little bit further, by a distance of dx .

$$dW = F_{gr} \cdot dx$$

At a distance of x from the surface, the distance between the centers of masses is $R + x$.



The gravitational force between the satellite and Earth is then

$$F_{gr} = \frac{mMG}{(x + R)^2}$$

and so the work required to lift the satellite by dx is

$$dW = F_{gr} \cdot dx = \frac{mMG}{(x + R)^2} dx$$

The total work required to lift the satellite from the surface to a height of H is then

$$\begin{aligned} W &= \int_0^H \frac{mMG}{(x+R)^2} dx = mMG \int_0^H \frac{1}{(x+R)^2} dx = mMG \left(-\frac{1}{x+R} \Big|_0^H \right) = mMG \left(-\frac{1}{H+R} - \left(-\frac{1}{0+R} \right) \right) \\ &= mMG \left(\frac{1}{R} - \frac{1}{H+R} \right) = 800 \text{ kg} \cdot 5.97 \times 10^{24} \text{ kg} \cdot 6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2} \left(\frac{1}{6370 \text{ m}} - \frac{1}{6370 \text{ m} + 5000 \text{ m}} \right) \\ &= 3.1856 \times 10^{17} \text{ N m}^2 \left(\frac{1}{6370 \text{ km}} - \frac{1}{11370 \text{ km}} \right) = 3.1856 \times 10^{17} \text{ N m}^2 \left(\frac{1}{6370000 \text{ m}} - \frac{1}{11370000 \text{ m}} \right) \\ &\approx 2.1992 \times 10^{10} \text{ N m} \end{aligned}$$

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E-mail questions or comments to mhidegkuti@ccc.edu.