Equation: $f(x) =$ ____________________

1. domain: ____________________

   range: ____________________

2. $y-$intercept: ____________________

   $x-$intercept(s): ____________________

3. horizontal asymptote(s): ____________________

   vertical asymptote(s): ____________________

4. increasing: ____________________

   decreasing: ____________________

5. relative maximum(s): ____________________

   absolute maximum(s): ____________________

   relative minimum(s): ____________________

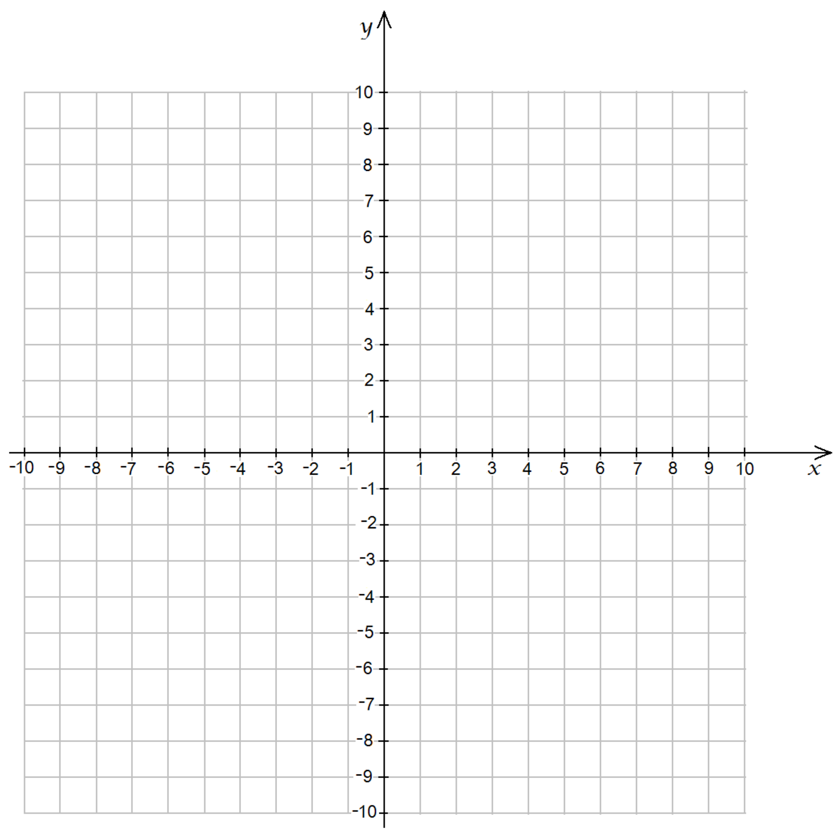
   absolute minimum(s): ____________________

6. continuous: ____________________

7. one-to-one: ____________________

8. end-behavior:
   $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to \infty} f(x)$

9. Graph



Definitions

1. The **domain** of a function is a non-empty set of elements to which we assign things.

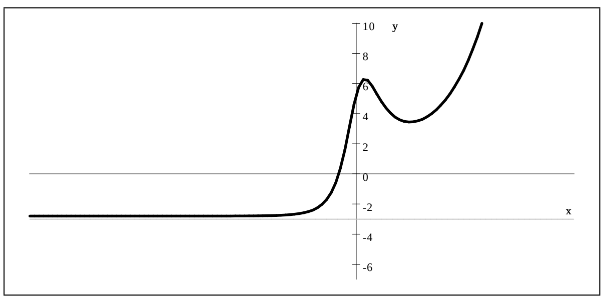
   The **range** of a function is the set of elements that we assign to elements of the domain.

2. The $y-$**intercept** of a function is the point where the graph of the function intersects the $y-$axis. A function can have at most one $y-$intercept.

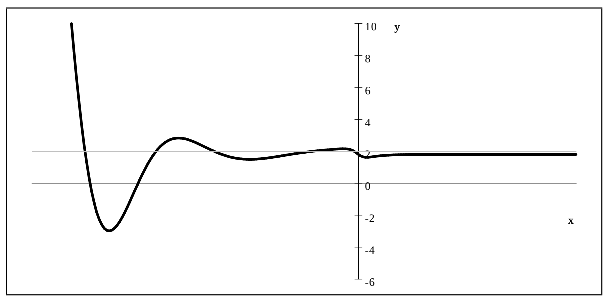
   The $x-$**intercept** of a function is the point where the graph of the function intersects the $x-$axis. A function can have several $x-$intercepts.

3. Asymptotes. At this point we will not rigorously define asymptotes yet, just provide with an intuitive idea. There are two types of asymptotes.

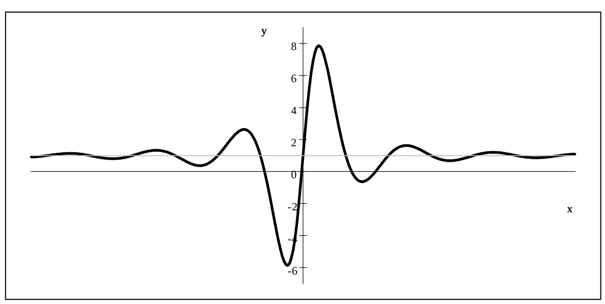
   A function can have at most two **horizontal asymptote**s. A horizontal asymptote is the graphical representation of the property that as the numbers in the domain get larger and larger (positive or negative), their assigned values get closer and closer to a fixed number.



As x gets larger and larger in the negative, $f(x)$ gets closer and closer to -3.



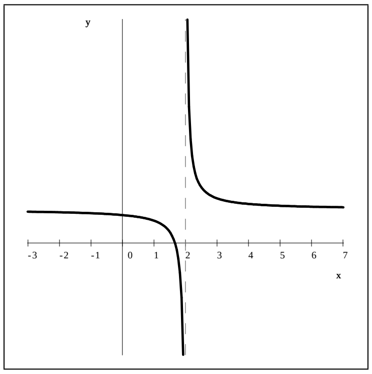
As x gets larger and larger in the positive, $f(x)$ gets closer and closer to 2.



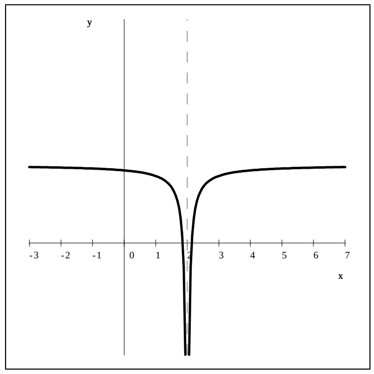
As x gets larger and larger, $f(x)$ gets closer and closer to 1.

The three examples above show different ways a function can approach a horizontal asymptote. The first picture shows a graph approaching a horizontal asympotote from above. The second graph approaches a horizontal asympotote from below. The third graph crosses it many times as it oscillates around it.

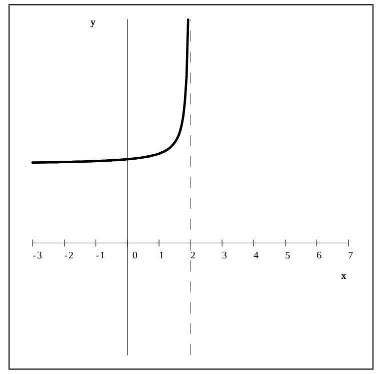
A **vertical asymptote** occurs when numbers in the domain close to a fixed number take extremely large values. A function can have many different vertical asymptotes. The function may behave in different ways at the left-hand side and at the right-hand side of the asymptote. Sometimes the function is not even defined on both sides of a vertical asymptote.



Vertical asymptote at $x = 2$



Vertical asymptote at $x = 2$



Vertical asymptote at $x = 2$

4. Definition: A function f is **increasing** on an interval I if for all a and b in I, if $a < b$, then $f(a) \leq f(b)$.

   Definition: A function f is **strictly increasing** on an interval I if for all a and b in I, if $a < b$, then $f(a) < f(b)$.

   Definition: A function f is **decreasing** on an interval I if for all a and b in I, if $a < b$, then $f(a) \geq f(b)$.

   Definition: A function f is **strictly decreasing** on an interval I if for all a and b in I, if $a < b$, then $f(a) > f(b)$.

5. Extremum is a common name for a maximum or a minimum.

   A function f has an **absolute maximum** at x_M if for all x in the domain, $f(x_M) \geq f(x)$.

   A function f has an **absolute minimum** at x_m if for all x in the domain, $f(x_m) \leq f(x)$.

   A function f has a **relative (or local) maximum** at x_M if there exists an open interval $I = (a, b)$ that contains x_M such that

   i) the function is defined on I and

   ii) if we restrict the function to I as its domain, then $(x_M, f(x_M))$ is an absolute maximum.

   A function f has a **relative (or local) minimum** at x_m if there exists an open interval $I = (a, b)$ that contains x_m such that

   i) the function is defined on I and

   ii) if we restrict the function to I as its domain, then $(x_m, f(x_m))$ is an absolute minimum.

6. Definition: A function f is **continuous** at a point c if $\lim\limits_{x \to c} f(x) = f(c)$.

   Definition: A function f is **continuous** on an open interval (a, b) if for all c with $a < c < b$, f is continuous at c.

   Definition: A function f is **continuous** on a closed interval $[a, b]$ if for all c with $a < c < b$, f is continuous at c, and $\lim\limits_{x \to a^-} f(x) = f(a)$ and $\lim\limits_{x \to b^+} f(x) = f(b)$.

7. Definition: A function f is **one-to-one (or injective)** if for all a and b in its domain, if $a \neq b$, then $f(a) \neq f(b)$.

   Alternative definition: A function f is **one-to-one (or injective)** if for all a and b in its domain, if $f(a) = f(b)$, then $a = b$.

8. The end-behavior of a function f describes the behavior of f for very large negative and very large positive numbers. In short,
$$\lim_{x\to-\infty} f(x) \quad \text{and} \quad \lim_{x\to\infty} f(x)$$

Sample Problems

Sketch the graph and give a complete analysis for each of the following functions.

1. $f(x) = \sqrt{x+1} - 2$
2. $f(x) = 10x - x^2 + 11$ where the domain is the closed interval $[3, 8]$
3. $f(x) = x^3 - 3x^2$ on $\mathbb{R}$

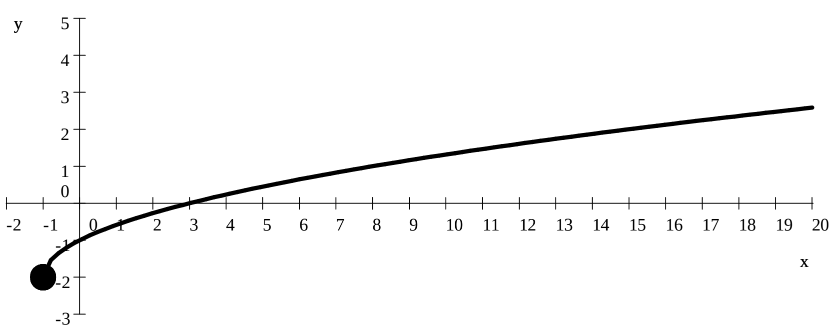
Sample Problems - Answers

1. $f(x) = \sqrt{x+1} - 2$

domain: $[-1, \infty)$
range: $[-2, \infty)$
$y-$intercept: $(0, -1)$
$x-$intercept: $(3, 0)$
no asymptotes

increasing on $(-1, \infty)$
decreasing nowhere
no relative maximum
no absolute maximum
no relative minimum
absolute minimum: $(-1, -2)$

continuous on $[-1, \infty)$
one-to-one
end-behavior:
$\lim_{x\to-\infty} f(x) =$ undefined
$\lim_{x\to\infty} f(x) = \infty$

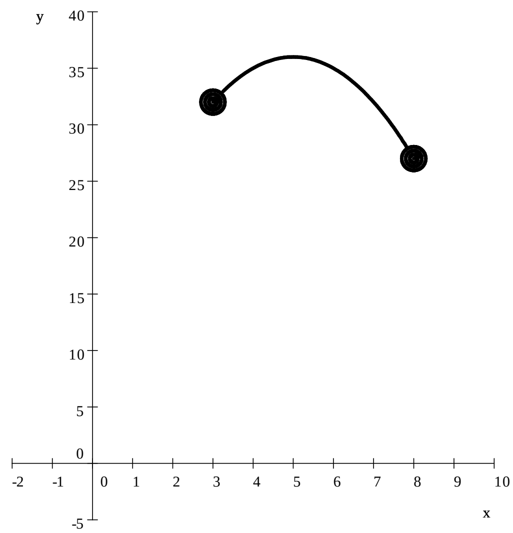


2. $f(x) = 10x - x^2 + 11$ where the domain is the closed interval $[3, 8]$.

domain: $[3, 8]$
range: $[27, 36]$
no intercepts
no asymptotes

increasing: on $(3, 5)$
decreasing: on $(5, 8)$
relative maximum: $(5, 36)$
absolute maximum: $(5, 36)$
no relative minimum
absolute minimum: $(8, 27)$

continuous on $[3, 8]$
not one-to-one
end-behavior:
$\lim_{x \to -\infty} f(x) =$ undefined
$\lim_{x \to \infty} f(x) =$ undefined



3. $f(x) = x^3 - 3x^2$ on $\mathbb{R}$

Since this is a cubic polynomial with a positive leading coefficient, the end-behavior is $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$.
We compute the intercepts: for the $y-$intercept, we compute $f(0) = 0$. For the $x-$intercept, we solve the equation $x^3 - 3x^2 = 0$

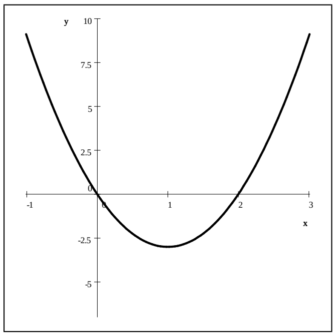
$$\begin{aligned} x^3 - 3x^2 &= 0 \\ x^2(x-3) &= 0 \quad \Longrightarrow \quad x_1 = 0 \quad \text{and } x_2 = 3 \end{aligned}$$

So the $x-$intercepts are $(0, 0)$ and $(3, 0)$.
Since this is a polynomial, there will be no vertical or horizontal asymptotes.

To find out where this function is increasing/decreasing, we need to see the sign of its derivative. We differentiate f

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

The derivative is a quadratic function, so its graph is a parabola. Its zeroes are at $x = 0$ and $x = 2$. Since its leading coefficient is positive, the parabola opens upward.



$f'(x)$

We see that f' is positive on $(-\infty, 0)$ and on $(2, \infty)$. This is where f is increasing. f' is negative on $(0, 2)$, which means that f is decreasing on $(0, 2)$.

At $x = 0$, the derivative f' changes sign from positive to negative. This implies that at $x = 0$, f changes from increasing to decreasing, and so there is a relative maximum there. At $x = 2$, the derivative f' changes sign from negative to positive. This implies that at $x = 2$, f changes behavior from decreasing to increasing, indicating is a relative minimum there.

We evaluate the function at $x = 0$ and 2 to find the $y-$coordinates: $f(0) = 0$ and $f(2) = -4$. And so f has a relative maximum: $(0, 0)$ and a relative minimum: $(2, -4)$.

What about absolute extrema? If we recall that the two end-behaviors are ∞ and $-\infty$, that means that there will not be an absolute minimum or maximum.

Because it is a polynomial function, f is continuous on its entire domain, $\mathbb{R}$. Since we already found two $x-$intercepts, f is clearly not one-to-one.

Based on all this, we can now graph the function and give its complete analysis:

domain: $\mathbb{R}$	increasing: on $(-\infty, 0)$ and on $(2, \infty)$	not one-to-one
range: $\mathbb{R}$	decreasing on: $(0, 2)$	continuous on $\mathbb{R}$
$y-$intercept: $(0, 0)$	relative maximum: $(0, 0)$	end-behavior:
$x-$intercepts: $(0, 0)$ and $(3, 0)$	no absolute maximum	$\lim\limits_{x\to-\infty} f(x) = -\infty$ and
no asymptotes	relative minimum: $(2, -4)$	$\lim\limits_{x\to\infty} f(x) = \infty$
	no absolute minimum