Suppose that an object is moving along a vertical line, and its vertical position is given by $L(t)$. The average velocity of the object between t_1 and t_2 is

$$v_{\text{av}} = \frac{L(t_2) - L(t_1)}{t_2 - t_1}$$

We define the instantenous velocity at t as the limit of the average velocities, where the time interval around t is getting smaller and smaller. In short, the instantaneous velocity at time t is the following limit (if this limit exists)

$$v(t) = \lim_{h\to 0} \frac{L(t+h) - L(t)}{t+h-t} = \lim_{h\to 0} \frac{L(t+h) - L(t)}{h}$$

Sample Problems

1. The location function of an object is $L(t) = t^2 - 3t$. Compute the instantaneous velocity of the object

   a) at $t = 7$ second b) at $t = 10$ second c) at t.

2. The location function of an object is $L(t) = t^3$. Compute the instantaneous velocity of the object

   a) at $t = 4$ second b) at t.

3. The location function of an object is $L(t) = \sqrt{t}$. Compute the instantaneous velocity of the object

   a) at $t = 49$ second b) at t

4. The location function of an object is $L(t) = \dfrac{1}{t}$. Compute the instantaneous velocity of the object

   a) at $t = 5$ second b) at t

Practice Problems

1. The location function of an object is $L(t) = -t^2 + t$. Compute the instantaneous velocity of the object

   a) at $t = 3$ second b) at $t = 4$ second c) at t

2. The location function of an object is $L(t) = t^4$. Compute the instantaneous velocity of the object

   a) at $t = 3$ second b) at t

   (Hint: you may need the following formula: $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$)

3. The location function of an object is $L(t) = \sqrt{2t+1}$. Compute the instantaneous velocity of the object

   a) at $t = 12$ second b) at t

4. The location function of an object is $L(t) = \dfrac{1}{3t+5}$. Compute the instantaneous velocity of the object

   a) at $t = 2$ second b) at t

5. (Enrichment) The location function of an object is given by $L(t) = 2t^3 - 15t^2$. When is the object moving upward?

Answers - Sample Problems

1. a) $v(7) = 11$ b) $v(10) = 17$ c) $v(t) = 2t - 3$
2. a) $v(4) = 48$ b) $v(t) = 3t^2$
3. a) $v(49) = \dfrac{1}{14}$ b) $v(t) = \dfrac{1}{2\sqrt{t}} = \dfrac{\sqrt{t}}{2t}$
4. a) $v(5) = -\dfrac{1}{25}$ b) $v(t) = -\dfrac{1}{t^2}$

Answers - Practice Problems

1. a) $v(3) = -5$ b) $v(4) = -7$ c) $v(t) = -2t + 1$
2. a) $v(3) = 108$ b) $v(t) = 4t^3$
3. a) $v(12) = \dfrac{1}{5}$ b) $L'(t) = \dfrac{1}{\sqrt{2t+1}}$
4. a) $v(2) = -\dfrac{3}{121}$ b) $v(t) = -\dfrac{3}{(3t+5)^2}$

Sample Problems - Solutions

1. The location function of an object is $L(t) = t^2 - 3t$.

   a) Compute the instantaneous velocity of the object at $t = 7$ second.

   Solution:

$$v_7 = \lim_{h \to 0} \frac{L(7+h) - L(7)}{h}$$

We compute first $L(7+h)$

$$L(7+h) = (7+h)^2 - 3(7+h) = h^2 + 14h + 49 - 21 - 3h = h^2 + 11h + 38$$

We also compute $L(7)$

$$L(7) = 7^2 - 3 \cdot 7 = 49 - 21 = 38$$

So now the velocity:

$$\begin{aligned} v_7 &= \lim_{h \to 0} \frac{L(7+h) - L(7)}{h} = \lim_{h \to 0} \frac{h^2 + 11h + 38 - 38}{h} = \lim_{h \to 0} \frac{h^2 + 11h}{h} = \lim_{h \to 0} \frac{\not{h}(h+11)}{\not{h}} \\ &= \lim_{h \to 0} (h + 11) = 11 \end{aligned}$$

So at $t = 7$, the velocity of the object is 11. In short, $v(7) = 11$.

b) Compute the instantaneous velocity of the object at $t = 10$ second.

Solution:
$$v(10) = \lim_{h\to 0} \frac{L(10+h) - L(10)}{h}$$

We compute first $L(10+h)$

$$L(10+h) = (10+h)^2 - 3(10+h) = h^2 + 20h + 100 - 30 - 3h = h^2 + 17h + 70$$

We also compute $L(10)$
$$L(10) = 10^2 - 3\cdot 10 = 100 - 30 = 70$$

So now the velocity:

$$\begin{aligned} v(10) &= \lim_{h\to 0} \frac{L(10+h) - L(10)}{h} = \lim_{h\to 0} \frac{h^2 + 17h + 70 - 70}{h} = \lim_{h\to 0} \frac{h^2 + 17h}{h} = \lim_{h\to 0} \frac{\not{h}(h+17)}{\not{h}} \\ &= \lim_{h\to 0} (h+17) = 17 \end{aligned}$$

So at $t = 10$, the velocity of the object is 17. In short, $v(10) = 17$.

c) Compute the instantaneous velocity of the object at t.

Solution: If we do that and we obtain an expression in terms of t, then we created a new function, the velocity function.
$$v(t) = \lim_{h\to 0} \frac{L(t+h) - L(t)}{h}$$

We compute first $L(t+h)$

$$L(t+h) = (t+h)^2 - 3(t+h) = h^2 + 2th + t^2 - 3t - 3h$$

So now the velocity:

$$\begin{aligned} v(10) &= \lim_{h\to 0} \frac{L(10+h) - L(10)}{h} = \lim_{h\to 0} \frac{\left(h^2 + 2th + t^2 - 3t - 3h\right) - \left(t^2 - 3t\right)}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 2th + t^2 - 3t - 3h - t^2 + 3t}{h} = \lim_{h\to 0} \frac{h^2 + 2th - 3h}{h} = \lim_{h\to 0} \frac{\not{h}(h + 2t - 3)}{\not{h}} \\ &= \lim_{h\to 0} (h + 2t - 3) = 2t - 3 \end{aligned}$$

So if an object's location is given by $L(t) = t^2 - 3t$, then its velocity at time t is $v(t) = 2t - 3$. If we look at this formula, $v(7) = 2\cdot 7 - 3 = 11$ and $v(10) = 2\cdot 10 - 3 = 17$ agrees with previous findings.

2. The location function of an object is $L(t) = t^3$.

a) Compute the instantaneous velocity of the object at $t = 4$ second.

Solution:

$$v(4) = \lim_{h\to 0} \frac{L(4+h) - L(4)}{h}$$

We compute first $L(4+h)$

$$L(4+h) = (4+h)^3 = 4^3 + 3\cdot 4^2 h + 3\cdot 4\cdot h^2 + h^3 = h^3 + 12h^2 + 48h + 64$$

We also compute $L(4) = 64$. So now the velocity:

$$\begin{aligned} v(4) &= \lim_{h\to 0} \frac{L(4+h) - L(4)}{h} = \lim_{h\to 0} \frac{h^3 + 12h^2 + 48h + 64 - 64}{h} = \lim_{h\to 0} \frac{h^3 + 12h^2 + 48h}{h} = \lim_{h\to 0} \frac{\cancel{h}\left(h^2 + 12h + 48\right)}{\cancel{h}} \\ &= \lim_{h\to 0} \left(h^2 + 12h + 48\right) = 48 \end{aligned}$$

So at $t = 4$, the velocity of the object is 48. In short, $v(4) = 48$.

b) Compute the instantaneous velocity of the object at t.

Solution: If we do that and we obtain an expression in terms of t, then we created a new function, the velocity function.

$$v(t) = \lim_{h\to 0} \frac{L(t+h) - L(t)}{h}$$

We compute first $L(t+h)$

$$L(t+h) = (t+h)^3 = t^3 + 3t^2h + 3th^2 + h^3$$

So now the velocity:

$$\begin{aligned} v(t) &= \lim_{h\to 0} \frac{L(t+h) - L(t)}{h} = \lim_{h\to 0} \frac{t^3 + 3t^2h + 3th^2 + h^3 - t^3}{h} = \lim_{h\to 0} \frac{3t^2h + 3th^2 + h^3}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}\left(3t^2 + 3th + h^2\right)}{\cancel{h}} = \lim_{h\to 0} \left(3t^2 + 3th + h^2\right) = 3t^2 \end{aligned}$$

So if an object's location is given by $L(t) = t^3$, then its velocity at time t is $v(t) = 3t^2$. If we look at this formula, $v(4) = 3\cdot 4^2 = 48$ agrees with previous findings.

3. The location function of an object is $L(t) = \sqrt{t}$.

a) Compute the instantaneous velocity of the object at $t = 49$ second.

Solution:

$$v(49) = \lim_{h\to 0} \frac{L(49+h) - L(49)}{h} = \lim_{h\to 0} \frac{\sqrt{49+h} - \sqrt{49}}{h} = \lim_{h\to 0} \frac{\sqrt{49+h} - 7}{h}$$

Since this is an indeterminate with radicals, we will use the conjugate of $\sqrt{49+h} - 7$.

$$\begin{aligned} v(49) &= \lim_{h\to 0} \frac{\sqrt{49+h} - 7}{h} = \lim_{h\to 0} \frac{\sqrt{49+h} - 7}{h} \cdot \frac{\sqrt{49+h} + 7}{\sqrt{49+h} + 7} \\ &= \lim_{h\to 0} \frac{49 + h - 49}{h\left(\sqrt{49+h} + 7\right)} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{49+h} + 7\right)} = \lim_{h\to 0} \frac{1}{\sqrt{49+h} + 7} = \frac{1}{14} \end{aligned}$$

So at $t = 49$, the velocity of the object is $\frac{1}{14}$. In short, $v(49) = \frac{1}{14}$.

b) Compute the instantaneous velocity of the object at t.

Solution:

$$v(t) = \lim_{h\to 0}\frac{L(t+h)-L(t)}{h} = \lim_{h\to 0}\frac{\sqrt{t+h}-\sqrt{t}}{h}$$

Since this is an indeterminate with radicals, we will use the conjugate of $\sqrt{t+h}-\sqrt{t}$.

$$\begin{aligned} v(49) &= \lim_{h\to 0}\frac{\sqrt{t+h}-\sqrt{t}}{h} = \lim_{h\to 0}\frac{\sqrt{t+h}-\sqrt{t}}{h}\cdot\frac{\sqrt{t+h}+\sqrt{t}}{\sqrt{t+h}+\sqrt{t}} \\ &= \lim_{h\to 0}\frac{t+h-t}{h\left(\sqrt{t+h}+\sqrt{t}\right)} = \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{t+h}+\sqrt{t}\right)} = \lim_{h\to 0}\frac{1}{\sqrt{t+h}+\sqrt{t}} = \frac{1}{2\sqrt{t}} \end{aligned}$$

So if an object's location is given by $L(t)=\sqrt{t}$, then its velocity at time t is $v(t)=\dfrac{1}{2\sqrt{t}}$. If we look at this formula, $v(49)=\dfrac{1}{2\sqrt{49}}=\dfrac{1}{14}$ agrees with previous findings.

4. The location function of an object is $L(t)=\dfrac{1}{t}$.

a) Compute the instantaneous velocity of the object at $t=5$ second.

Solution:

$$\begin{aligned} v(5) &= \lim_{h\to 0}\frac{L(5+h)-L(5)}{h} = \lim_{h\to 0}\frac{\frac{1}{5+h}-\frac{1}{5}}{h} = \lim_{h\to 0}\frac{\frac{5-(5+h)}{5(5+h)}}{h} \\ &= \lim_{h\to 0}\left(\frac{1}{h}\cdot\frac{5-5-h}{5(5+h)}\right) = \lim_{h\to 0}\frac{-h}{5h(5+h)} = \lim_{h\to 0}\frac{-1}{5(5+h)} = -\frac{1}{25} \end{aligned}$$

So at $t=5$, the velocity of the object is $-\dfrac{1}{25}$. In short, $v(5)=-\dfrac{1}{25}$. The negative sign here indicates that the object is moving downward at $t=5$ second.

b) Compute the instantaneous velocity of the object at t.

Solution:

$$\begin{aligned} v(t) &= \lim_{h\to 0}\frac{L(t+h)-L(t)}{h} = \lim_{h\to 0}\frac{\frac{1}{t+h}-\frac{1}{t}}{h} = \lim_{h\to 0}\frac{\frac{t-(t+h)}{t(t+h)}}{h} \\ &= \lim_{h\to 0}\left(\frac{1}{h}\cdot\frac{t-t-h}{t(t+h)}\right) = \lim_{h\to 0}\frac{-h}{th(t+h)} = \lim_{h\to 0}\frac{-1}{t(t+h)} = -\frac{1}{t^2} \end{aligned}$$

So if an object's location is given by $L(t)=\dfrac{1}{t}$, then its velocity at time t is $v(t)=-\dfrac{1}{t^2}$. If we look at this formula, $v(5)=-\dfrac{1}{25}$ agrees with previous findings.

