We are about to learn about an important integration technique called substitution or u-subsitution. We will also see that this technique in pratice is a reversal of the chain rule. The following examples can be solved by u-substitution, but they are not complicated enough to really need it. On the other hand, this technique might motivate the idea of substitution. This technique works only for a specific type of a function, namely composed functions in which the inner function is linear.

The technique: Guess the answer. Differentiate your guess. **If we are off only by a constant multiplyer**, that can be fixed by a pre-emptive division.

Example 1. Find $\displaystyle\int \cos\left(3x+\frac{\pi}{2}\right)dx$.

Solution: We know that the antiderivative of cosine is sine. So it is a reasonable guess that the answer should be something like $\sin(3x+\pi)$. Let us differentiate our guess.

$$\left(\sin\left(3x+\frac{\pi}{2}\right)\right)' = \cos\left(3x+\frac{\pi}{2}\right)\cdot 3$$

So, the antiderivative is close, but not perfect. Our guess overlooked the multiplyer of 3 that pops out of the chain rule when differentiating. We can fix that: our answer should be then $\dfrac{1}{3}\sin\left(3x+\dfrac{\pi}{2}\right)$ When we differentiate, we get the derivative $\cos\left(3x+\dfrac{\pi}{2}\right)$. Therefore, $\displaystyle\int \cos\left(3x+\frac{\pi}{2}\right)dx = \frac{1}{3}\sin\left(3x+\frac{\pi}{2}\right)+C$.

The technique works because derivatives behave nicely with constant multiplyers: $(c\cdot f)' = c\cdot f'$ for c real number and f differentiable function. Linear functions are the only functions to have a constant derivative.

Example 2. Find $\displaystyle\int \frac{1}{1-x}dx$.

Solution: The smaller the chain rule, the more dangerous it is. The constant multiplyer caused by the cahin rule is -1. It is a common error to overlook it. The antiderivative of $\dfrac{1}{x}$ is $\ln|x|$. Our first guess will be off by -1. The correct answer is then $\displaystyle\int \frac{1}{1-x}dx = -\ln|x-1|+C$ or $-\ln|1-x|+C$.

Example 3. Find $\displaystyle\int \frac{6x+1}{2x-3}dx$.

Solution: We can use long division to dividde $6x+1$ by $2x-3$. What we will do here, is pretty much the same thing but using a different notation. $\dfrac{6x}{2x}=3$, and $3(2x-3)=6x-9$. But the numerator is $6x+1$. So we will smuggle in what we need for the division.

$$\frac{6x+1}{2x-3} = \frac{6x-9+9+1}{2x-3} = \frac{6x-9}{2x-3}+\frac{10}{2x-3} = 3+\frac{10}{2x-3}$$

$$\int\frac{6x+1}{2x-3}dx = \int\left(3+\frac{10}{2x-3}\right)dx = 3x+10\cdot\frac{1}{2}\ln|2x-3|+C = 3x+5\ln|2x-3|+C$$

Example 4. Find $\displaystyle\int \cos^2 x\,dx$.

Solution: We will use the double angle formula for cosine. $\cos 2x = 2\cos^2 x - 1$, then $\cos^2 x = \dfrac{1+\cos 2x}{2}$. This will be much easier to integrate.

$$\int\cos^2 x\,dx = \int\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)dx = \frac{1}{2}x+\frac{1}{4}\sin 2x + C$$

Practice Problems

Compute each of the following antiderivatives.

1. $\displaystyle\int e^{5x}\,dx$

2. $\displaystyle\int \sin 2\theta \; d\theta$

3. $\displaystyle\int (2y+3)^{10} \; dy$

4. $\displaystyle\int \frac{1}{7x-3}\,dx$

5. $\displaystyle\int (2-x)^{3} \; dx$

6. $\displaystyle\int e^{8t-1}\,dt$

7. $\displaystyle\int \cos(\pi\theta) \; d\theta$

8. $\displaystyle\int \frac{x+2}{x-1}\,dx$

9. $\displaystyle\int \cos\left(2\alpha-\frac{\pi}{4}\right) \; d\alpha$

10. $\displaystyle\int \frac{1}{\left(\frac{1}{3}x-1\right)^{5}}\,dx$

11. $\displaystyle\int \left(-2^{5t}+1\right) \; dt$

12. $\displaystyle\int \sqrt{5m+1}\,dm$

13. $\displaystyle\int \frac{1}{\sqrt{4a+1}}\,da$

Answers

1. $\dfrac{1}{5}e^{5x}+C$ 2. $-\dfrac{1}{2}\cos 2\theta+C$ 3. $\displaystyle\int \frac{(2y+3)^{11}}{22}+C$ 4. $\dfrac{1}{7}\ln|7x-3|+C$ 5. $-\dfrac{1}{4}(2-x)^4+C$

6. $\dfrac{1}{8}e^{8t-1}+C$ 7. $\dfrac{1}{\pi}\sin\pi\theta+C$ 8. $x+3\ln|x-1|+C$ 9. $\dfrac{1}{2}\sin\left(2\alpha-\dfrac{\pi}{4}\right)+C$

10. $-\dfrac{3}{4\left(\dfrac{1}{3}x-1\right)^4}+C$ 11. $-\dfrac{2^{5t}}{5\ln 2}+t+C$ 12. $\dfrac{2}{15}(5m+1)^{3/2}+C$ 13. $\dfrac{1}{2}\sqrt{4a+1}+C$

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