Sample Problems

1. Consider the straight line $y = \frac{1}{2}x$ on the interval $[0, 6]$. Suppose we rotate the line around the $x-$axis. Find the volume of the object we obtain by
   (a) first writing a Riemann sum estimating the volume
   (b) turning the Riemann sum into an integral and evaluating it.
   (c) Repeat the previous computations using the function $f(x) = mx$ on the interval $[0, h]$, and r denoting the value of $f(h) = mh$.

2. We are standing on the top of a 60 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 250 N and the rope's unit weight is $2\frac{\mathrm{N}}{\mathrm{m}}$.
   (a) Find a Riemann sum expressing the amount of work it takes to lift the bucket from the ground to the top of the building.
   (b) Turn the Riemann sum into an integral and evaluate it to find the work.

3. A cylindrical tank has height 10 m and base radius 3 m. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m^3 of water weighs 10 000 N, i.e. the 'density' of water is $\delta = 10\,000\frac{\mathrm{N}}{\mathrm{m}^3}$. How much work does it require to pump all water out of the tank?
   (a) Find a Riemann sum expressing the work.
   (b) Turn the Riemann sum into an integral and evaluate it.

4. A tank, shaped like a cone has height 10 m and base radius 3 m. It is placed so that the circular part is upward. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m^3 of water weighs 10 000 N, i.e. the 'density' of water is $\delta = 10\,000\,\frac{\mathrm{N}}{\mathrm{m}^3}$. How much work does it require to pump all water out of the tank?
   (a) Find a Riemann sum expressing the work.
   (b) Turn the Riemann sum into an integral and evaluate it.

Sample Problems - Answers

1. a) $V_{\text{total}} = \sum_{i=0}^{n-1} \frac{\pi\ x_i^2}{4} \Delta x$ or $V_{\text{total}} = \sum_{i=1}^{n} \frac{\pi\ x_i^2}{4} \Delta x$ b) 18π c) $\frac{\pi r^2\ h}{3}$

2. a) $W_{\text{total}} = \sum_{i=0}^{n-1} (370 - 2x_i)\, \Delta x$ or $W_{\text{total}} = \sum_{i=1}^{n} (370 - 2x_i)\, \Delta x$ b) $18\,600\,\mathrm{J}$

3. a) $W_{\text{total}} = \sum_{i=0}^{n-1} \left(\delta\pi r^2\right) (10 - x_i)\, \Delta x$ or $W_{\text{total}} = \sum_{i=1}^{n} \left(\delta\pi r^2\right) (10 - x_i)\, \Delta x$

   b) $4500\,000\pi\ \mathrm{J} \approx 1.413\,72 \times 10^7\ \mathrm{J}$

4. a) $W_{\text{total}} = \sum_{i=0}^{n-1} \frac{9\delta\pi x_i^2}{100}\ (10 - x_i)\, \Delta x$ or $W_{\text{total}} = \sum_{i=1}^{n} \frac{9\delta\pi x_i^2}{100}\ (10 - x_i)\, \Delta x$ b) $750\ 000\ \pi\ \mathrm{J} \approx 2.356\,2 \times 10^6\ \mathrm{J}$

Sample Problems - Solutions

1. Consider the straight line $y = \frac{1}{2}x$ on the interval $[0, 6]$. Suppose we rotate the line around the $x-$axis. Find the volume of the object we obtain by

   a) first writing a Riemann sum estimating the volume
   Solution: Let us partition the interval $[0, 6]$ into n equal intervals by $0 = x_0,\ x_1,\ x_{2,\dots},x_n = 6$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$. We will approximate the function $f(x) = \frac{1}{2}x$ as constant $f(x_i)$ over the entire interval $[x_i, x_{i+1}]$. Then the volume of a slice will be that of a cylinder with radius $f(x_i)$ and height Δx. The volume of the cylindrical slice is

$$V_i = \pi r^2 h = \pi (f(x_i))^2 \Delta x = \pi \left(\frac{x_i}{2}\right)^2 \Delta x = \frac{\pi\ x_i^2}{4}\Delta x$$

   The total volume is

$$V_{\text{total}} = \sum_{i=0}^{n-1} V_i = \sum_{i=0}^{n-1} \frac{\pi\ x_i^2}{4}\Delta x$$

   b) turning the Riemann sum into an integral and evaluating it.
   Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} V_{\text{total}} &= \lim_{n\to\infty}\sum_{i=0}^{n-1} V_i = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{\pi\ x_i^2}{4}\Delta x = \int_0^6 \left(\frac{\pi\ x^2}{4}\right)dx = \frac{\pi}{4}\int_0^6 x^2 dx = \frac{\pi}{4}\left(\frac{x^3}{3}\right)\Bigg|_0^6 \\ &= \frac{\pi}{4}\left(\frac{6^3}{3} - \frac{0^3}{3}\right) = \frac{\pi}{4}(72) = 18\pi \end{aligned}$$

   b) Repeat the previous computations using the function $f(x) = mx$ on the interval $[0, h]$, and r denoting the value of $f(h) = mh$.

$$V_i = \pi r^2 h = \pi (f(x_i))^2 \Delta x = \pi (mx_i)^2 \Delta x = \pi m^2 x_i^2\ \Delta x$$

   and thus the volume is

$$\begin{aligned} V_{\text{total}} &= \lim_{n\to\infty}\sum_{i=0}^{n-1} V_i = \lim_{n\to\infty}\sum_{i=0}^{n-1}\pi m^2 x_i^2\ \Delta x = \int_0^h \left(\pi m^2 x^2\right)dx = \pi m^2\int_0^h x^2 dx = \pi m^2\left(\frac{x^3}{3}\right)\Bigg|_0^h \\ &= \pi m^2\left(\frac{h^3}{3} - \frac{0^3}{3}\right) = \frac{\pi m^2 h^3}{3} = \frac{\pi\left(m^2h^2\right)h}{3} = \frac{\pi r^2\ h}{3} \end{aligned}$$

2. We are standing on the top of a 60 m tall building. We have to lift a bucket from the ground, using a rope. The bucket weighs 250 N and the rope's unit weight is $2\frac{\text{N}}{\text{m}}$.

   a) Find a Riemann sum expressing the amount of work it takes to lift the bucket from the ground to the top of the building.
   Solution: Let x represent the height of the bucket. Let us partition the interval $[0, 60]$ into n equal intervals by $0 = x_0,\ x_1,\ x_{2,\dots},x_n = 60$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$. We will approximate the height of the bucket to be x_i over the entire interval $[x_i, x_{i+1}]$. Then the force we have to exert while lifting the bucket from height x_i to x_{i+1} is

$$F_i = F_{\text{bucket}} + F_{\text{rope}} = 250 + 2(60 - x_i) = 370 - 2x_i$$

Then the work we have to invest when lifting the bucket from height x_i to x_{i+1} is

$$W_i = F_i \cdot s_i = (370 - 2x_i)(x_{i+1} - x_i) = (370 - 2x_i)\,\Delta x$$

The total work is then

$$W_{\text{total}} = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} (370 - 2x_i)\,\Delta x$$

b) Turn the Riemann sum into an integral and evaluate it to find the work.
Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned} W_{\text{total}} &= \lim_{n\to\infty} \sum_{i=0}^{n-1} (370 - 2x_i)\,\Delta x = \int_0^{60} (370 - 2x_i)\,dx = 370x - x^2\Big|_0^{60} \\ &= \left(370\,(60) - (60)^2\right) - \left(370\,(0) - (0)^2\right) = 22\,200 - 3600 = 18\,600 \end{aligned}$$

(a) or, the same computation with the units, quite elegant,

$$\begin{aligned} W_{\text{total}} &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(370\,\text{N} - 2\frac{\text{N}}{\text{m}} x_i\right) \Delta x = \int_0^{60\,\text{m}} \left(370\,\text{N} - 2\frac{\text{N}}{\text{m}} x\right) dx \\ &= \left(370\,\text{N}x - x^2\frac{\text{N}}{\text{m}}\right)\Big|_{0\,\text{m}}^{60\,\text{m}} \\ &= \left[370\,\text{N}\,(60\,\text{m}) - (60\,\text{m})^2\,\frac{\text{N}}{\text{m}}\right] - \left[370\,\text{N}\,(0\,\text{m}) - (0\,\text{m})^2\,\frac{\text{N}}{\text{m}}\right] \\ &= 22\,200\,\text{N m} - 3600\,\text{N m} = 18\,600\,\text{N m} \\ &= 18\,600\,\text{J} \end{aligned}$$

3. A cylindrical tank has height 10 m and base radius 3 m. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that $1\,\text{m}^3$ of water weighs 10 000 N, i.e. the 'density' of water is $\delta = 10\,000\frac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?
a) Find a Riemann sum expressing the work.
Solution: Let x represent the water level. Let us partition the interval $[0, 10]$ into n equal intervals by $0 = x_0,\ x_1,\ x_2, \ldots, x_n = 10$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$. We will approximate the water level to be x_i over the entire interval $[x_i, x_{i+1}]$. Then the force we have to exert when pumping out the water between heights x_i and x_{i+1} is the weight of a cylindrical 'slice' with volume

$$\begin{aligned} V_i &= \pi r^2 \Delta x \quad \text{the weight is then} \quad F_i = \delta V_i \\ F_i &= \delta \pi r^2 \Delta x \end{aligned}$$

The water needs to be lifted to the top of the tank, and so

$$s_i = 10 - x_i$$

Then the work we have to invest when lifting the slice of water from height x_i to 10 is

$$W_i = F_i \cdot s_i = \left(\delta\pi r^2 \Delta x\right)(10 - x_i) = \left(\delta\pi r^2\right)(10 - x_i)\,\Delta x$$

The total work is then is

$$W_{\text{total}} = \sum_{i=0}^{n-1} W_i = \sum_{i=0}^{n-1} F_i \cdot s_i = \sum_{i=0}^{n-1} \left(\delta\pi r^2\right)(10 - x_i)\,\Delta x$$

b) Turn the Riemann sum into an integral and evaluate it.
Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned}
W_{\text{total}} &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left(\delta\pi r^2\right)(10-x_i)\,\Delta x = \int_0^{10}\left(\delta\pi r^2\right)(10-x)\,dx = \delta\pi r^2\int_0^{10}(10-x)\,dx \\
&= 10\,000\pi\,(9)\int_0^{10}(10-x)\,dx = 90\,000\pi\,\left(10x-\frac{x^2}{2}\right)\Big|_0^{10} \\
&= 90\,000\pi\,\left[\left(10\,(10)-\frac{(10)^2}{2}\right)-\left(10\,(0)-\frac{(0)^2}{2}\right)\right] \\
&= 90\,000\pi\,(50) = 4500\,000\pi \approx 1.413\,72\times 10^7\ \text{J}
\end{aligned}$$

4. A tank, shaped like a cone has height 10 m and base radius 3 m. It is placed so that the circular part is upward. It is full of water, and we have to pump it all out by a pipe that is always leveled at the surface of the water. Assume that 1 m^3 of water weighs 10 000 N, i.e. the 'density' of water is $\delta = 10\,000\ \dfrac{\text{N}}{\text{m}^3}$. How much work does it require to pump all water out of the tank?

a) Find a Riemann sum expressing the work.
Solution: Let x represent the water level. Let us partition the interval $[0,10]$ into n equal intervals by $0 = x_0,\ x_1,\ x_{2,\dots},x_n = 10$, where the points are equally placed, and let Δx denote the length of a subinterval $[x_i, x_{i+1}]$. We will approximate the water level to be x_i over the entire interval $[x_i, x_{i+1}]$. Then the force we have to exert when pumping out the water between heights x_i and x_{i+1} is the weight of a cylindrical 'slice' with volume $\pi r_i^2 h$ where $h = \Delta x$ and r_i is the radius at height x_i. Clearly $\tan\alpha = \dfrac{r_i}{x_i} = \dfrac{3}{10}$ which gives us that $r_i = \dfrac{3}{10}x_i$

$$\begin{aligned}
V_i &= \pi r_i^2\Delta x = \pi\left(\frac{3}{10}x_i\right)^2\Delta x = \frac{9\pi x_i^2}{100}\Delta x \text{ the weight is then } F_i = \delta V_i \\
F_i &= \frac{9\delta\pi x_i^2}{100}\Delta x
\end{aligned}$$

The water needs to be lifted to the top of the tank, and so

$$s_i = 10 - x_i$$

Then the work we have to invest when lifting the slice of water from height x_i to 10 is

$$W_i = F_i\cdot s_i = \left(\frac{9\delta\pi x_i^2}{100}\Delta x\right)(10-x_i) = \frac{9\delta\pi x_i^2}{100}\ (10-x_i)\,\Delta x$$

The total work is then is

$$W_{\text{total}} = \sum_{i=0}^{n-1}W_i = \sum_{i=0}^{n-1}F_i\cdot s_i = \sum_{i=0}^{n-1}\frac{9\delta\pi x_i^2}{100}\ (10-x_i)\,\Delta x$$

b) Turn the Riemann sum into an integral and evaluate it.

Solution: As n approaches infinity, the Riemann sum approaches the integral

$$\begin{aligned}
W_{\text{total}} &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{9\delta\pi x_i^2}{100}\ (10-x_i)\,\Delta x = \int\limits_0^{10}\frac{9\delta\pi x^2}{100}\ (10-x)\,dx \\
&= \frac{9\delta\pi}{100}\int\limits_0^{10} x^2\,(10-x)\,dx = \frac{9\,(10\,000)\,\pi}{100}\int\limits_0^{10} x^2\,(10-x)\,dx \\
&= 900\pi\ \int_0^{10}\left(10x^2-x^3\right)dx \\
&= 900\pi\ \left(\frac{10x^3}{3}-\frac{x^4}{4}\right)\Bigg|_0^{10} \\
&= 900\pi\left[\left(\frac{10\,(10)^3}{3}-\frac{10^4}{4}\right)-\left(\frac{0^3}{3}-\frac{0^4}{4}\right)\right] \\
&= 900\pi\left(\frac{10^4}{3}-\frac{10^4}{4}\right) = 900\pi\left(\frac{1}{12}\right)10^4 \\
&= 750\,000\pi
\end{aligned}$$

