

Sample Problems

1. Consider the solid obtained by revolving the region under the graph of $y = \sqrt{x}$ on $[0, 4]$ about the x -axis.
 - a) Let us use the partition $\{x_0, x_1, \dots, x_n\}$ and approximate the function as constant within each subinterval. Let us rotate this approximation about the x -axis. Write an expression for V_i , the volume of the i th small disk.
 - b) Write a Riemann sum approximating the volume of the solid.
 - c) Find an integral expressing the volume and evaluate the integral.
2. Consider the solid obtained by revolving the region under the graph of $y = \frac{1}{2}x$ on $[0, 10]$ about the x -axis.
 - a) Write a Riemann sum expressing the volume.
 - b) Find an integral expressing the volume and evaluate the integral.
3. Consider the solid obtained by revolving the region under the graph of $y = mx$ on $[0, h]$ about the x -axis.
 - a) Write a Riemann sum expressing the volume.
 - b) Find an integral expressing the volume and evaluate the integral.
 - c) This object is a cone with height h and base radius $r = mh$. Express the volume computed in part b) in terms of r and h .
4. Compute the volume of the solid obtained by revolving the region under the graph of $y = \sqrt{R^2 - x^2}$ on $[-R, R]$ about the x -axis.
5. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = \sqrt{x}$, $y = 2$, and $x = 0$ about the y -axis.
6. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^3$, $y = 8$, and $x = 0$ about the y -axis.
7. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = \arctan x$, $y = \frac{\pi}{4}$, and $x = 0$ about the y -axis.
8. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^2$, $y = 0$, and $x = 2$ about the x -axis.
9. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^2$, $x = 0$, and $y = 4$ about the y -axis.

Practice Problems

1. Find the volume of the solid obtained by rotating the region bounded by $y = \sin x$, $y = 0$ between $x = 0$ and $x = \pi$ about the x -axis.
2. Compute the volume of the solid obtained by rotating the region bounded by $y = \sqrt{\ln x}$, $y = 0$, $x = 1$, and $x = e$ about the x -axis.
3. Compute the volume of the solid obtained by rotating the region bounded by $y = e^{-x}$, $y = 0$, $x = 0$, and $x = 1$ about the x -axis.
4. Compute the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{x}$, $x = 1$, and $y = 0$ about the x -axis. (This will be an improper integral.)
5. Compute the volume of the solid obtained by rotating the region bounded by $y = \operatorname{sech} x$, $y = 0$, $x = 0$ and $x = 1$ about the x -axis.
6. a) Compute the volume of the solid obtained by rotating the region bounded by $y = \frac{x}{3}$, $y = 0$, and $x = 6$ about the x -axis.
b) Compute the volume of the solid obtained by rotating the region bounded by $y = \frac{x}{3}$, $y = 2$, and $x = 0$ about the y -axis.
7. a) Compute the volume of the solid obtained by rotating the region bounded by $y = \sqrt[4]{x}$, $y = 0$, and $x = 16$ about the x -axis.
b) Compute the volume of the solid obtained by rotating the region bounded by $y = \sqrt[4]{x}$, $y = 2$, and $x = 0$ about the y -axis.
8. Find the volume common to two spheres, each with radius r , if the center of each sphere lies on the surface of the other sphere.

Sample Problems - Answers

$$1.) \quad \text{a)} \quad V_i = \pi [f(x_i)]^2 \Delta x_i \quad \text{b)} \quad \sum_{i=1}^n V_i = \sum_{i=1}^n \pi [f(x_i)]^2 \Delta x_i \quad \text{c)} \quad 8\pi$$

$$2.) \quad \text{a)} \quad \sum_{i=1}^n \pi [f(x_i)]^2 \Delta x_i \quad \text{b)} \quad \frac{250}{3}\pi \quad 3.) \quad \text{a)} \quad \sum_{i=1}^n \pi [f(x_i)]^2 \Delta x_i \quad \text{b)} \quad \frac{1}{3}\pi m^2 h^3 \quad \text{c)} \quad \frac{1}{3}\pi r^2 h$$

$$4.) \quad \frac{4}{3}\pi R^3 \quad 5.) \quad \frac{32}{5}\pi \quad 6.) \quad \frac{96}{5}\pi \quad 7.) \quad \pi - \frac{\pi^2}{4} \quad 8.) \quad \frac{32}{5}\pi \quad 9.) \quad 8\pi$$

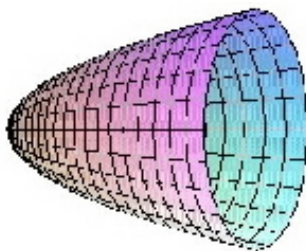
Practice Problems - Answers

$$1.) \quad \frac{\pi^2}{2} \quad 2.) \quad \pi \quad 3.) \quad \frac{\pi}{2} \left(1 - \frac{1}{e^2}\right) \quad 4.) \quad \pi \quad 5.) \quad \frac{e^2 - 1}{e^2 + 1}\pi \quad 6.) \quad \text{a)} \quad 8\pi \quad \text{b)} \quad 24\pi$$

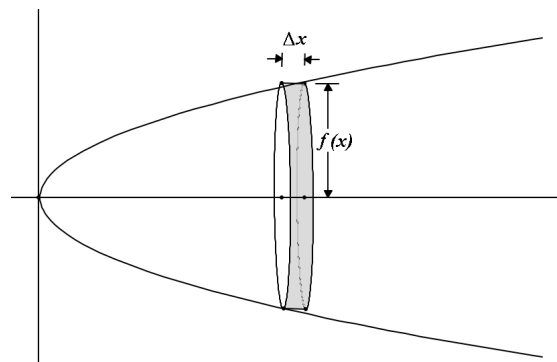
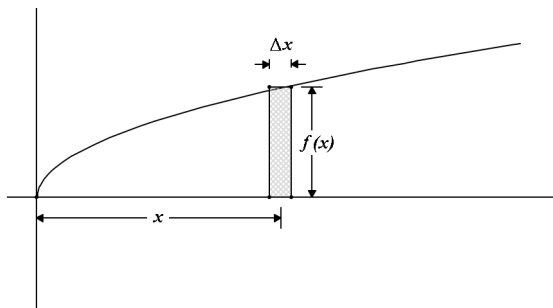
$$7.) \quad \text{a)} \quad \frac{128}{3}\pi \quad \text{b)} \quad \frac{512}{9}\pi \quad 8.) \quad \frac{5}{12}\pi r^3$$

Sample Problems - Solutions

1. Consider the solid obtained by revolving the region under the graph of $y = \sqrt{x}$ on $[0, 4]$ about the x -axis.



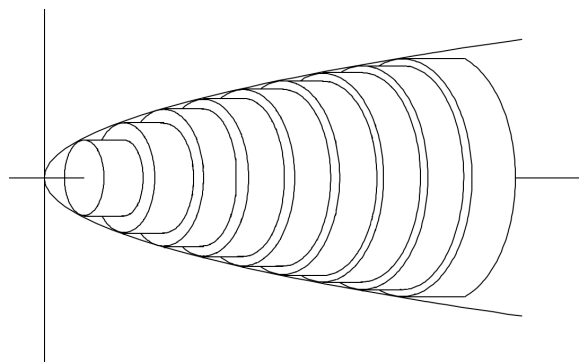
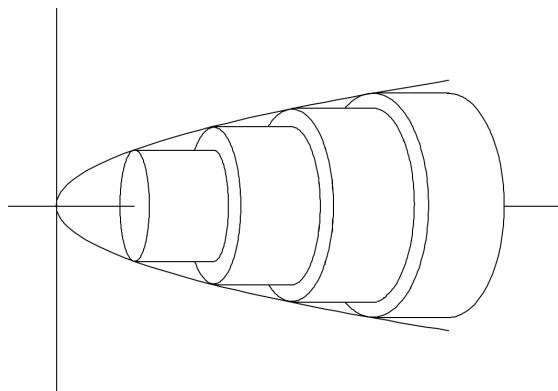
- a) Let us use the partition $\{x_0, x_1, \dots, x_n\}$ and approximate the function as constant within each subinterval. Let us rotate this approximation about the x -axis. Write an expression for V_i , the volume of the i th small disk.



Solution: One thin disk is created by rotating the rectangle shown on the picture above. The rectangle has height $f(x)$ and width Δx . When we rotate this rectangle, we obtain a cylinder with radius $f(x)$ and height Δx . A cylinder's volume is $V = \pi r^2 h$. In this case,

$$V_i = \pi [f(x_i)]^2 \Delta x_i$$

- b) Write a Riemann sum approximating the volume of the solid.



$$\sum_{i=1}^n V_i = \sum_{i=1}^n \pi [f(x_i)]^2 \Delta x_i$$

c) Find an integral expressing the volume and evaluate the integral.

Solution: As n approaches infinity, the Riemann sum becomes a definite integral.

$$\begin{aligned} V &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \pi [f(x_i)]^2 \Delta x_i = \int_0^4 \pi [f(x)]^2 dx = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x dx = \pi \left. \frac{x^2}{2} \right|_0^4 = \frac{\pi}{2} (4^2 - 0^2) \\ &= \frac{\pi}{2} (16) = 8\pi \end{aligned}$$

2. Consider the solid obtained by revolving the region under the graph of $y = \frac{1}{2}x$ on $[0, 10]$ about the x -axis.

a) Write a Riemann sum expressing the volume.

Solution: Let x be fixed. One thin disk is created by rotating the rectangle shown on the picture above.

A cylinder's volume is $V = \pi r^2 h$. In this case, $\Delta V = \pi [f(x_i)]^2 \Delta x_i$. The Riemann sum is then

$$\sum_{i=1}^n \pi [f(x_i)]^2 \Delta x_i$$

b) Find an integral expressing the volume and evaluate the integral.

Solution: As n approaches infinity, the Riemann sum becomes a definite integral.

$$\begin{aligned} V &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \pi [f(x_i)]^2 \Delta x_i = \int_0^{10} \pi [f(x)]^2 dx = \int_0^{10} \pi \left(\frac{x}{2}\right)^2 dx = \int_0^{10} \frac{\pi}{4} x^2 dx = \frac{\pi}{4} \int_0^{10} x^2 dx \\ &= \frac{\pi}{4} \left(\frac{x^3}{3} \right) \Big|_0^{10} = \frac{\pi}{12} (10^3 - 0^3) = \frac{\pi}{12} (1000) = \frac{250}{3} \pi \end{aligned}$$

3. Consider the solid obtained by revolving the region under the graph of $y = mx$ on $[0, h]$ about the x -axis.

a) Write a Riemann sum expressing the volume.

Solution: Let x be fixed. The i th thin disk is created by rotating a rectangle with width Δx_i and height $f(x_i)$. The volume of such a cylinder is $V_i = \pi [f(x_i)]^2 \Delta x_i$. The Riemann sum is then

$$\sum_{i=1}^n \pi [f(x_i)]^2 \Delta x_i$$

b) Find an integral expressing the volume and evaluate the integral.

Solution: As n approaches infinity, the Riemann sum becomes a definite integral.

$$\begin{aligned} V &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \pi [f(x_i)]^2 \Delta x_i = \int_0^h \pi [f(x)]^2 dx = \int_0^h \pi m^2 x^2 dx = \pi m^2 \int_0^h x^2 dx = \pi m^2 \left(\frac{x^3}{3} \right) \Big|_0^h \\ &= \frac{\pi m^2}{3} (h^3 - 0^3) = \frac{1}{3} \pi m^2 h^3 \end{aligned}$$

c) This object is a cone with height h and base radius $r = mh$. Express the volume computed in part b) in terms of r and h .

Solution:

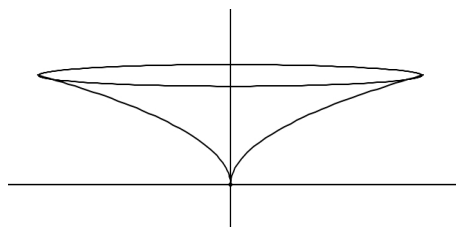
$$V = \frac{1}{3} \pi m^2 h^3 = \frac{1}{3} \pi (m^2 h^2) h = \frac{1}{3} \pi (mh)^2 h = \frac{1}{3} \pi r^2 h$$

4. Compute the volume of the solid obtained by revolving the region under the graph of $y = \sqrt{R^2 - x^2}$ on $[-R, R]$ about the x -axis.

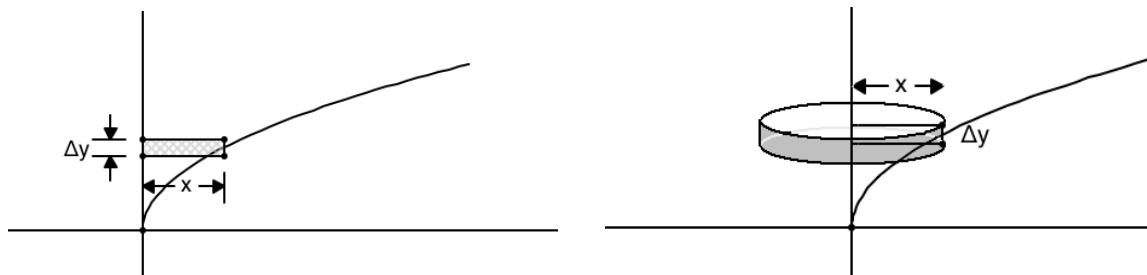
Solution:

$$\begin{aligned}
 V_i &= \pi [f(x_i)]^2 \Delta x_i & \text{Riemann sum} &= \sum_{i=1}^n \pi [f(x_i)]^2 \Delta x_i \\
 V &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \pi [f(x_i)]^2 \Delta x_i = \int_{-R}^R \pi [f(x)]^2 dx = \pi \int_{-R}^R (\sqrt{R^2 - x^2})^2 dx = \pi \int_{-R}^R R^2 - x^2 dx \\
 &= 2\pi \int_0^R R^2 - x^2 dx = 2\pi \left(R^2 x - \frac{x^3}{3} \right) \Big|_0^R = 2\pi \left(R^3 - \frac{R^3}{3} \right) = 2\pi \left(\frac{2}{3} R^3 \right) = \frac{4}{3} \pi R^3
 \end{aligned}$$

5. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = \sqrt{x}$, $y = 2$, and $x = 0$ about the y -axis.



Solution: We will again rotate rectangles to obtain cylinders but this time we rotate them about the y -axis.



We will need to express everything in terms of y . If $y = \sqrt{x}$, then $x = y^2$.

The volume of one disk is $V_i = \pi x_i^2 \Delta y_i$. then the Riemann sum is $\sum_{i=1}^n \pi x_i^2 \Delta y_i = \sum_{i=1}^n \pi [f^{-1}(y_i)]^2 \Delta y_i$. The volume is the limit of the Riemann sum as n approaches infinity.

$$\begin{aligned}
 V &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \pi x_i^2 \Delta y_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \pi [f^{-1}(y_i)]^2 \Delta y_i = \int_0^2 \pi [f^{-1}(y)]^2 dy = \int_0^2 \pi (y^2)^2 dy = \pi \int_0^2 y^4 dy \\
 &= \pi \frac{y^5}{5} \Big|_0^2 = \frac{\pi}{5} (2^5 - 0^5) = \frac{32}{5} \pi
 \end{aligned}$$

6. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^3$, $y = 8$, and $x = 0$ about the y -axis.

Solution: If $y = x^3$, then $x = \sqrt[3]{y}$. We will need to express everything in terms of y .

The volume of one disk is $V_i = \pi x_i^2 \Delta y_i$. Then the entire object's volume is

$$\begin{aligned} V &= \int_0^8 \pi x^2 dy = \int_0^8 \pi (\sqrt[3]{y})^2 dy = \pi \int_0^8 y^{2/3} dy = \pi \left(\frac{3}{5} \right) y^{5/3} \Big|_0^8 = \frac{3}{5} \pi (8^{5/3} - 0^{5/3}) \\ &= \frac{3}{5} \pi (32) = \frac{96}{5} \pi \end{aligned}$$

7. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = \arctan x$, $y = \frac{\pi}{4}$, and $x = 0$ about the y -axis.

Solution: If $y = \arctan x$, then $x = \tan y$. We will need to express everything in terms of y .

The volume of one disk is $V_i = \pi x^2 dy$. Then the entire object's volume is

$$V = \int_0^{\pi/4} \pi x^2 dy = \int_0^{\pi/4} \pi (\tan y)^2 dy = \pi \int_0^{\pi/4} \tan^2 y dy$$

Recall that

$$\int \tan^2 y dy = \int \sec^2 y - 1 dy = \tan y - y + C$$

$$\begin{aligned} \pi \int_0^{\pi/4} \tan^2 y dy &= \pi (\tan y - y) \Big|_0^{\pi/4} = \pi \left[\left(\tan \frac{\pi}{4} - \frac{\pi}{4} \right) - (\tan 0 - 0) \right] = \pi \left[\left(1 - \frac{\pi}{4} \right) - 0 \right] \\ &= \pi \left(1 - \frac{\pi}{4} \right) = \pi - \frac{\pi^2}{4} \end{aligned}$$

8. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^2$, $y = 0$, and $x = 2$ about the x -axis.

Solution:

$$\int_0^2 \pi [f(x)]^2 dx = \int_0^2 \pi (x^2)^2 dx = \pi \int_0^2 x^4 dx = \pi \frac{x^5}{5} \Big|_0^2 = \frac{\pi}{5} (2^5 - 0^5) = \frac{32}{5} \pi$$

9. Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^2$, $x = 0$, and $y = 4$ about the y -axis.

Solution: $x = \sqrt{y}$

$$\int_0^4 \pi (x)^2 dy = \int_0^4 \pi (\sqrt{y})^2 dy = \pi \int_0^4 y dy = \pi \frac{y^2}{2} \Big|_0^4 = \frac{\pi}{2} (4^2 - 0^2) = 8\pi$$