

Sample Problems

Compute each of the following integrals. Please note that $\arcsin x$ is the same as $\sin^{-1} x$ and $\arctan x$ is the same as $\tan^{-1} x$

1. $\int x e^x dx$

5. $\int \arcsin x dx$

9. $\int \cos^2 x dx$

2. $\int x \cos x dx$

6. $\int \arctan x dx$

10. $\int x^2 e^{-3x} dx$

3. $\int x e^{-4x} dx$

7. $\int e^x \sin x dx$

11. $\int \frac{x^3}{(x^2 + 2)^2} dx$

4. $\int \ln x dx$

8. $\int \sin^2 x dx$

Practice Problems

1. $\int x e^{2x} dx$

6. $\int x^2 2^x dx$

11. $\int x \sin 10x dx$

2. $\int x e^{-3x} dx$

7. $\int x \cos x dx$

12. $\int_1^9 \frac{\ln x}{\sqrt{x}} dx$

3. $\int_0^{\ln 2} x e^{-3x} dx$

8. $\int x^2 \cos x dx$

13. $\int_1^{\infty} \frac{\ln x}{x^7} dx$

4. $\int_0^{\infty} x e^{-3x} dx$

9. $\int x \ln x dx$

5. $\int x 2^x dx$

10. $\int x^5 \ln x dx$

14. $\int_0^{\pi/4} x \sin 2x dx$

Sample Problems - Answers

1.) $x e^x - e^x + C$ 2.) $x \sin x + \cos x + C$ 3.) $-\frac{1}{16} e^{-4x} - \frac{1}{4} x e^{-4x} + C$ 4.) $x \ln x - x + C$

5.) $x \arcsin x + \sqrt{1-x^2} + C$ 6.) $x \arctan x - \frac{1}{2} \ln(x^2 + 1) + C$ 7.) $\frac{1}{2} e^x (\sin x - \cos x) + C$

8.) $\frac{1}{2} (-\sin x \cos x + x) + C$ 9.) $\frac{1}{2} (x + \sin x \cos x) + C$ 10.) $-e^{-3x} \left(\frac{1}{3} x^2 + \frac{2}{9} x + \frac{2}{27} \right)$

11.) $\frac{1}{2} \ln(x^2 + 2) + \frac{1}{x^2 + 2} + C$

Practice Problems - Answers

1.) $\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$ 2.) $-\frac{1}{9} e^{-3x} - \frac{1}{3} x e^{-3x} + C$ 3.) $\frac{7}{72} - \frac{1}{24} \ln 2$ 4.) $\frac{1}{9}$

5.) $\frac{2^x}{\ln 2} \left(x - \frac{1}{\ln 2} \right) + C$ 6.) $\frac{2^x}{\ln 2} \left(x^2 - \frac{2x}{\ln 2} + \frac{2}{\ln^2 2} \right) + C$ 7.) $x \sin x + \cos x + C$

8.) $x^2 \sin x + 2x \cos x - 2 \sin x + C$ 9.) $\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C$ 10.) $\frac{1}{6} x^6 \ln x - \frac{1}{36} x^6 + C$

11.) $\frac{1}{100} \sin 10x - \frac{1}{10} x \cos 10x + C$ 12.) $6 \ln 9 - 8$ 13.) $\frac{1}{36}$ 14.) $\frac{1}{4}$

Sample Problems - Solutions

Please note that $\arcsin x$ is the same as $\sin^{-1} x$ and $\arctan x$ is the same as $\tan^{-1} x$.

1. $\int x e^x dx$

Solution: We will integrate this by parts, using the formula

$$\int f'g = fg - \int fg'$$

Let $g(x) = x$ and $f'(x) = e^x$. Then we obtain g' and f by differentiation and integration.

$f(x) = e^x$	$g(x) = x$
$f'(x) = e^x$	$g'(x) = 1$

$$\begin{aligned} \int f'g &= fg - \int fg' \quad \text{becomes} \\ \int x e^x dx &= x e^x - \int e^x dx = \boxed{x e^x - e^x + C} \end{aligned}$$

We should check our result by differentiating the answer. Indeed,

$$(x e^x - e^x + C)' = e^x + x e^x - e^x = x e^x$$

and so our answer is correct.

2. $\int x \cos x dx$

Solution: Let $g(x) = x$ and $f'(x) = \cos x$. Then we obtain g' and f by differentiation and integration.

$f(x) = \sin x$	$g(x) = x$
$f'(x) = \cos x$	$g'(x) = 1$

$$\begin{aligned} \int f'g &= fg - \int fg' \quad \text{becomes} \\ \int x \cos x dx &= x \sin x - \int \sin x dx = x \sin x - (-\cos x) = \boxed{x \sin x + \cos x + C} \end{aligned}$$

We should check our result by differentiating the answer. Indeed,

$$(x \sin x + \cos x + C)' = \sin x + x \cos x - \sin x = x \cos x$$

and so our answer is correct.

$$3. \int x e^{-4x} dx$$

Solution: Let $g(x) = x$ and $f'(x) = e^{-4x}$. Then we obtain g' and f by differentiation and integration. To compute $f(x)$, we will use substitution. Let $u = -4x$ then $du = -4dx$ and so $dx = \frac{du}{-4}$.

$$f(x) = \int e^{-4x} dx = \int e^u \frac{du}{-4} = -\frac{1}{4} \int e^u du = -\frac{1}{4} e^u + C = -\frac{1}{4} e^{-4x} + C$$

We will choose $C = 0$ and so $f(x) = -\frac{1}{4} e^{-4x}$.

$f(x) = -\frac{1}{4} e^{-4x}$	$g(x) = x$
$f'(x) = e^{-4x}$	$g'(x) = 1$

$$\begin{aligned} \int f'g &= fg - \int fg' \quad \text{becomes} \\ \int x e^{-4x} dx &= -\frac{1}{4} x e^{-4x} - \int -\frac{1}{4} e^{-4x} dx = -\frac{1}{4} x e^{-4x} + \frac{1}{4} \int e^{-4x} dx = -\frac{1}{4} x e^{-4x} + \frac{1}{4} \left(-\frac{1}{4} e^{-4x} \right) + C \\ &= \boxed{-\frac{1}{4} x e^{-4x} - \frac{1}{16} e^{-4x} + C} \end{aligned}$$

We check our result by differentiating the answer.

$$\begin{aligned} \left(-\frac{1}{4} x e^{-4x} - \frac{1}{16} e^{-4x} + C \right)' &= \\ &= -\frac{1}{4} (x e^{-4x})' - \frac{1}{16} (e^{-4x})' = -\frac{1}{4} (e^{-4x} + x(-4e^{-4x})) - \frac{1}{16} (-4e^{-4x}) \\ &= -\frac{1}{4} e^{-4x} + x e^{-4x} + \frac{1}{4} e^{-4x} = x e^{-4x} \end{aligned}$$

and so our answer is correct.

$$4. \int \ln x dx$$

Solution: Let $g(x) = \ln x$ and $f'(x) = 1$. Then we obtain g' and f by differentiation and integration.

$f(x) = x$	$g(x) = \ln x$
$f'(x) = 1$	$g'(x) = \frac{1}{x}$

$$\begin{aligned} \int f'g &= fg - \int fg' \quad \text{becomes} \\ \int \ln x dx &= x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int 1 dx = \boxed{x \ln x - x + C} \end{aligned}$$

We check our result by differentiating the answer.

$$(x \ln x - x + C)' = \ln x + x \cdot \frac{1}{x} - 1 = \ln x$$

and so our answer is correct.

5. $\int \arcsin x \, dx$

Solution: Let $g(x) = \arcsin x$ and $f'(x) = 1$. Then we obtain g' and f by differentiation and integration.

$f(x) = x$	$g(x) = \arcsin x$
$f'(x) = 1$	$g'(x) = \frac{1}{\sqrt{1-x^2}}$

$$\int f'g = fg - \int fg' \quad \text{becomes}$$

$$\int \arcsin x \, dx = x \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx$$

We compute the integral $\int \frac{x}{\sqrt{1-x^2}} \, dx$ by substitution. Let $u = 1 - x^2$. Then $du = -2x \, dx$ and so $dx = \frac{du}{-2x}$.

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}} \, dx &= \int \frac{x}{\sqrt{u}} \frac{du}{-2x} = -\frac{1}{2} \int \frac{1}{\sqrt{u}} \, du = -\frac{1}{2} \int u^{-1/2} \, du \\ &= -\frac{1}{2} \frac{u^{1/2}}{\frac{1}{2}} + C = -\sqrt{u} + C = -\sqrt{1-x^2} + C \end{aligned}$$

Thus the entire integral is

$$\int \arcsin x \, dx = x \arcsin x - \left(-\sqrt{1-x^2}\right) + C = \boxed{x \arcsin x + \sqrt{1-x^2} + C}$$

We check our result by differentiating the answer.

$$\begin{aligned} \left(x \arcsin x + \sqrt{1-x^2} + C\right)' &= \\ &= (x \arcsin x)' + \left((1-x^2)^{1/2}\right)' = \arcsin x + x \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{2} (1-x^2)^{-1/2} (-2x) \\ &= \arcsin x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \arcsin x \end{aligned}$$

and so our answer is correct.

6. $\int \arctan x \, dx$

Solution: Let $g(x) = \arctan x$ and $f'(x) = 1$. Then we obtain g' and f by differentiation and integration.

$f(x) = x$	$g(x) = \arctan x$
$f'(x) = 1$	$g'(x) = \frac{1}{x^2+1}$

$$\int f'g = fg - \int fg' \quad \text{becomes}$$

$$\int \arctan x \, dx = x \arctan x - \int x \cdot \frac{1}{x^2+1} \, dx = x \arctan x - \int \frac{x}{x^2+1} \, dx$$

We compute the integral $\int \frac{x}{x^2+1} dx$ by substitution. Let $u = x^2 + 1$. Then $du = 2x dx$ and so $dx = \frac{du}{2x}$.

$$\int \frac{x}{x^2+1} dx = \int \frac{x}{u} \frac{du}{2x} = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln (x^2 + 1) + C$$

Thus the entire integral is

$$\int \arctan x dx = \boxed{x \arctan x - \frac{1}{2} \ln (x^2 + 1) + C}$$

We check our result by differentiating the answer.

$$\begin{aligned} \left(x \arctan x - \frac{1}{2} \ln (x^2 + 1) + C \right)' &= \\ &= (x \arctan x)' - \frac{1}{2} (\ln (x^2 + 1))' = \arctan x + x \cdot \frac{1}{x^2 + 1} - \frac{1}{2} \frac{1}{x^2 + 1} (2x) \\ &= \arctan x + \frac{x}{x^2 + 1} - \frac{x}{x^2 + 1} = \arctan x \end{aligned}$$

so our answer is correct.

7. $\int e^x \sin x dx$

Solution: This is an interesting application of integration by parts. Let M denote the integral $\int e^x \sin x dx$.

Solution: Let $g(x) = \sin x$ and $f'(x) = e^x$ (Notice that because of the symmetry, $g(x) = e^x$ and $f'(x) = \sin x$ would also work.) We obtain g' and f by differentiation and integration.

$f(x) = e^x$	$g(x) = \sin x$
$f'(x) = e^x$	$g'(x) = \cos x$

$$\begin{aligned} \int f'g &= fg - \int fg' \text{ becomes} \\ \int e^x \sin x dx &= e^x \sin x - \int e^x \cos x dx \end{aligned}$$

It looks like our method produced a new integral, $\int e^x \cos x dx$ that also requires integration by parts. We proceed: let $g(x) = \cos x$ and $f'(x) = e^x$. We obtain g' and f by differentiation and integration.

$f(x) = e^x$	$g(x) = \cos x$
$f'(x) = e^x$	$g'(x) = -\sin x$

$$\begin{aligned} \int f'g &= fg - \int fg' \text{ becomes} \\ \int e^x \cos x dx &= e^x \cos x - \int e^x (-\sin x) dx = e^x \cos x + \int e^x \sin x dx \end{aligned}$$

$$\text{Thus } \int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$$

Now the result contains the original integral, $\int e^x \sin x$. At this point, it looks like we are getting nowhere because we are going in circles. However, this is not the case. Recall that we denote $\int e^x \sin x$ by M . Let us review the computation again:

$$\begin{aligned}\int e^x \sin x \, dx &= e^x \sin x - \int e^x \cos x \, dx \\ \int e^x \sin x \, dx &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right) \\ \int e^x \sin x \, dx &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx\end{aligned}$$

This is the same as

$$M = e^x \sin x - e^x \cos x - M$$

This is an equation that we can solve for M .

$$\begin{aligned}2M &= e^x \sin x - e^x \cos x \\ M &= \frac{1}{2}e^x (\sin x - \cos x)\end{aligned}$$

Thus the answer is $\boxed{\frac{1}{2}e^x (\sin x - \cos x) + C}$. We check our result by differentiation.

$$\begin{aligned}\left(\frac{1}{2}e^x (\sin x - \cos x)\right)' &= \\ &= \frac{1}{2}(e^x)'(\sin x - \cos x) + \frac{1}{2}e^x (\sin x - \cos x)' = \frac{1}{2}e^x (\sin x - \cos x) + \frac{1}{2}e^x (\cos x + \sin x) \\ &= \frac{1}{2}e^x (\sin x - \cos x + \sin x + \cos x) = \frac{1}{2}e^x (2 \sin x) = e^x \sin x\end{aligned}$$

so our answer is correct.

8. $\int \sin^2 x \, dx$

Solution: Note that this integral can be easily solved using substitution. This is because of the double angle formula for cosine, $\cos 2x = 1 - 2\sin^2 x \implies \sin^2 x = \frac{1 - \cos 2x}{2}$. This solution can be found on our substitution handout. But at the moment, we will use this interesting application of integration by parts as seen in the previous problem.

Let M denote the integral $\int \sin^2 x \, dx$. Let $g(x) = \sin x$ and $f'(x) = \sin x$. Then we obtain g' and f by differentiation and integration.

$f(x) = -\cos x$	$g(x) = \sin x$
$f'(x) = \sin x$	$g'(x) = \cos x$

$$\begin{aligned}\int f'g &= fg - \int fg' \quad \text{becomes} \\ \int \sin^2 x \, dx &= -\sin x \cos x - \int (-\cos x) \cos x \, dx = -\sin x \cos x + \int \cos^2 x \, dx \\ &= -\sin x \cos x + \int 1 - \sin^2 x \, dx = -\sin x \cos x + \int 1 \, dx - \int \sin^2 x \, dx \\ &= -\sin x \cos x + x - \int \sin^2 x \, dx\end{aligned}$$

We obtained

$$\begin{aligned}\int \sin^2 x \, dx &= -\sin x \cos x + x - \int \sin^2 x \, dx \quad \text{or} \\ M &= -\sin x \cos x + x - M \quad \text{we solve for } M \\ 2M &= -\sin x \cos x + x \\ M &= \frac{1}{2}(-\sin x \cos x + x) + C\end{aligned}$$

So our answer is $\boxed{\frac{1}{2}(-\sin x \cos x + x) + C}$. We check our result by differentiating the answer.

$$\begin{aligned}\left(\frac{1}{2}(-\sin x \cos x + x) + C\right)' &= \\ &= \left(\frac{1}{2}(-\sin x \cos x + x) + C\right)' = \frac{1}{2}(-\sin x(-\sin x) + (-\cos x)(\cos x) + 1) \\ &= \frac{1}{2}(\sin^2 x - \cos^2 x + 1) = \frac{1}{2}\left(\sin^2 x + \underbrace{1 - \cos^2 x}_{\sin^2 x}\right) = \frac{1}{2}(2\sin^2 x) = \sin^2 x\end{aligned}$$

so our answer is correct.

9. $\int \cos^2 x \, dx$

Solution: We do not need to integrate by parts (although it is good practice)

$$\begin{aligned}\int \cos^2 x \, dx &= \int 1 - \sin^2 x \, dx = \int 1 \, dx - \int \sin^2 x \, dx = x - \frac{1}{2}(-\sin x \cos x + x) + C \\ &= \frac{1}{2}\sin x \cos x + \frac{1}{2}x + C = \boxed{\frac{1}{2}(x + \sin x \cos x) + C}\end{aligned}$$

We check our result by differentiating the answer.

$$\left(\frac{1}{2}(x + \sin x \cos x) + C\right)' = \frac{1}{2}(1 + \cos^2 x - \sin^2 x) = \frac{1}{2}\left(\underbrace{1 - \sin^2 x}_{\cos^2 x} + \cos^2 x\right) = \frac{1}{2}(2\cos^2 x) = \cos^2 x$$

so our answer is correct.

$$10. \int x^2 e^{-3x} dx$$

Solution: We will need to integrate by parts twice. First, let $f'(x) = e^{-3x}$ and $g(x) = x^2$. Then

$f(x) = -\frac{1}{3}e^{-3x}$	$g(x) = x^2$
$f'(x) = e^{-3x}$	$g'(x) = 2x$

$$\begin{aligned} \int f'g &= fg - \int fg' \text{ becomes} \\ \int x^2 e^{-3x} dx &= -\frac{1}{3}e^{-3x}(x^2) - \int \left(-\frac{1}{3}e^{-3x}\right) 2x dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} dx \end{aligned}$$

and we can compute $\int x e^{-3x} dx$ by integrating by parts. Let $f'(x) = e^{-3x}$ and $g(x) = x$. Then

$f(x) = -\frac{1}{3}e^{-3x}$	$g(x) = x$
$f'(x) = e^{-3x}$	$g'(x) = 1$

$$\begin{aligned} \int f'g &= fg - \int fg' \text{ becomes} \\ \int x e^{-3x} dx &= -\frac{1}{3}e^{-3x}(x) - \int \left(-\frac{1}{3}e^{-3x}\right) dx = -\frac{1}{3}x e^{-3x} + \frac{1}{3} \int e^{-3x} dx \\ &= -\frac{1}{3}x e^{-3x} + \frac{1}{3} \left(-\frac{1}{3}e^{-3x}\right) + C = -\frac{1}{3}x e^{-3x} - \frac{1}{9}e^{-3x} + C \end{aligned}$$

This is the result we need to compute the integral $\int x^2 e^{-3x} dx$. So far we had this much:

$$\int x^2 e^{-3x} dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} dx$$

to this we substitute our result $\int x e^{-3x} dx = -\frac{1}{3}x e^{-3x} - \frac{1}{9}e^{-3x} + C$:

$$\begin{aligned} \int x^2 e^{-3x} dx &= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3} \left(-\frac{1}{3}x e^{-3x} - \frac{1}{9}e^{-3x} + C_1\right) \\ &= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}x e^{-3x} - \frac{2}{27}e^{-3x} + C \end{aligned}$$

Our result might look nicer if we factor out $-e^{-3x}$ or $-\frac{1}{27}e^{-3x}$. Then the final answer is

$$\boxed{-e^{-3x} \left(\frac{1}{3}x^2 + \frac{2}{9}x + \frac{2}{27}\right) + C} \text{ or } \boxed{-\frac{1}{27}e^{-3x} (9x^2 + 6x + 2) + C}.$$

We check via differentiation:

$$\begin{aligned} f'(x) &= \left(-\frac{1}{27}e^{-3x} (9x^2 + 6x + 2)\right)' = -\frac{1}{27} (-3e^{-3x} (9x^2 + 6x + 2) + e^{-3x} (18x + 6)) \\ &= -\frac{1}{27} (e^{-3x} (-27x^2 - 18x - 6) + e^{-3x} (18x + 6)) \\ &= -\frac{1}{27} e^{-3x} (-27x^2 - 18x - 6 + 18x + 6) = x^2 e^{-3x} \end{aligned}$$

and so our solution is correct.

$$11. \int \frac{x^3}{(x^2 + 2)^2} dx$$

Solution: this integral can be computed using at least three different methods: substitution (try $u = x^2 + 2$) or partial fractions or integration by parts. We will present integration by parts here.

First, let $f'(x) = \frac{x}{(x^2 + 2)^2}$ and $g(x) = x^2$. To compute f , we need to integrate $\frac{x}{(x^2 + 2)^2}$. We can do

that by using substitution: Let $u = x^2 + 2$. Then $du = 2x dx$ and so $dx = \frac{du}{2x}$. So

$$\int \frac{x}{(x^2 + 2)^2} dx = \int \frac{x}{u^2} \frac{du}{2x} = \frac{1}{2} \int \frac{1}{u^2} du = \frac{1}{2} \left(-\frac{1}{u} \right) + C = -\frac{1}{2(x^2 + 2)} + C$$

Thus if $f'(x) = \frac{x}{(x^2 + 2)^2}$, then $f(x) = -\frac{1}{2(x^2 + 2)}$. Proceeding with the integration by parts, we write

$f(x) = -\frac{1}{2(x^2 + 2)}$	$g(x) = x^2$
$f'(x) = \frac{x}{(x^2 + 2)^2}$	$g'(x) = 2x$

$$\begin{aligned} \int f'g &= fg - \int fg' \quad \text{becomes} \\ \int \frac{x}{(x^2 + 2)^2} (x^2) dx &= -\frac{1}{2(x^2 + 2)} (x^2) - \int -\frac{1}{2(x^2 + 2)} (2x) dx \\ &= -\frac{x^2}{2(x^2 + 2)} + \int \frac{x}{x^2 + 2} dx \end{aligned}$$

and this second integral can be computed using the same substitution:

Let $w = x^2 + 2$. Then $dw = 2x dx$ and so $dx = \frac{dw}{2x}$

$$\int \frac{x}{x^2 + 2} dx = \int \frac{x}{w} \frac{dw}{2x} = \frac{1}{2} \int \frac{1}{w} dw = \frac{1}{2} \ln |w| + C = \frac{1}{2} \ln (x^2 + 2) + C$$

and so the entire integral is then

$$\int \frac{x^3}{(x^2 + 2)^2} dx = -\frac{x^2}{2(x^2 + 2)} + \int \frac{x}{x^2 + 2} dx = \boxed{-\frac{x^2}{2(x^2 + 2)} + \frac{1}{2} \ln (x^2 + 2) + C}$$

We check via differentiation:

$$\begin{aligned} f'(x) &= \left(-\frac{x^2}{2(x^2 + 2)} + \frac{1}{2} \ln (x^2 + 2) + C \right)' \\ &= -\frac{1}{2} \left(\frac{2x(x^2 + 2) - x^2(2x)}{(x^2 + 2)^2} \right) + \frac{1}{2} \frac{1}{x^2 + 2} (2x) \\ &= -\left(\frac{x(x^2 + 2) - x^3}{(x^2 + 2)^2} \right) + \frac{x}{x^2 + 2} = -\left(\frac{x^3 + 2x - x^3}{(x^2 + 2)^2} \right) + \frac{x}{x^2 + 2} \\ &= -\frac{2x}{(x^2 + 2)^2} + \frac{x(x^2 + 2)}{(x^2 + 2)^2} = \frac{-2x + x^3 + 2x}{(x^2 + 2)^2} = \frac{x^3}{(x^2 + 2)^2} \end{aligned}$$

and so our solution is correct.

For more documents like this, visit our page at <https://teaching.martahidegkuti.com> and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.