Sample Problems

Compute each of the following integrals. Assume that a and b are positive.

1. $\displaystyle\int e^{-4x}\,dx$
2. $\displaystyle\int_0^8 e^{-4x}\,dx$
3. $\displaystyle\int \left(x^2-2\right)\left(x^3-6x\right)^{207}\,dx$
4. $\displaystyle\int e^{\cos x}\sin x\,dx$
5. $\displaystyle\int_0^{\pi/2} e^{\cos x}\sin x\,dx$
6. $\displaystyle\int \frac{3x}{\left(x^2+1\right)^7}\,dx$
7. $\displaystyle\int \frac{12x^3}{3x^4+1}\,dx$
8. $\displaystyle\int (-3x+4)\,e^{-3x^2+8x}\,dx$
9. $\displaystyle\int_0^2 (-3x+4)\,e^{-3x^2+8x}\,dx$
10. $\displaystyle\int \frac{x+5}{x^2+1}\,dx$
11. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$
12. $\displaystyle\int \cos x\sin^5 x\,dx$
13. $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$
14. $\displaystyle\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx$
15. $\displaystyle\int_0^3 \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$
16. $\displaystyle\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\,dx$
17. $\displaystyle\int_0^1 \frac{x+1}{x^2+1}\,dx$
18. $\displaystyle\int_0^1 xe^{-3x^2}\,dx$
19. $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx$
20. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$
21. $\displaystyle\int \frac{x^3}{\left(x^2+2\right)^2}\,dx$
22. $\displaystyle\int \frac{1}{x^2+9}\,dx$
23. $\displaystyle\int \frac{1}{a^2x^2+b^2}\,dx$
24. $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$
25. $\displaystyle\int_0^{\pi/3} \sin 2x\sin x\,dx$
26. $\displaystyle\int 42\cos x\sin x\left(\sin x+1\right)^5\,dx$
27. $\displaystyle\int_e^{e^2} \frac{1}{x\ln\sqrt{x}}\,dx$
28. $\displaystyle\int \tan x\,dx$
29. $\displaystyle\int \sec x\,dx$
30. $\displaystyle\int \frac{x^2}{\sqrt{1-\left(x^3-1\right)^2}}\,dx$
31. $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$
32. $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$
33. $\displaystyle\int_4^5 \frac{1}{4x\ln x\left[\ln\left(\ln x\right)\right]^3}dx$
34. $\displaystyle\int_0^{\pi/3} \sin x\left(\cos x-\cos^3 x\right)\,dx$
35. $\displaystyle\int \sin^2 x\,dx$
36. $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx$
37. $\displaystyle 2\int_{-r}^{r} \sqrt{r^2-x^2}\,dx$

Sample Problems - Answers

1.) $-\frac{1}{4}e^{-4x}+C$ 2.) $\frac{1}{4}-\frac{1}{4}e^{-32}$ 3.) $\frac{1}{624}\left(x^3-6x\right)^{208}+C$ 4.) $-e^{\cos x}+C$ 5.) $e-1$

6.) $-\frac{1}{4\left(x^2+1\right)^6}+C$ 7.) $\ln\left(3x^4+1\right)+C$ 8.) $\frac{1}{2}e^{-3x^2+8x}+C$ 9.) $\frac{1}{2}e^4-\frac{1}{2}$

10.) $5\tan^{-1}x+\frac{1}{2}\ln\left(x^2+1\right)+C$ 11.) $\ln\left(e^x+1\right)+C$ 12.) $\frac{1}{6}\sin^6 x+C$ 13.) $\frac{2}{3}(x+1)^{3/2}-2(x+1)^{1/2}+C$

14.) 36 15.) $\ln\left(e^3+\frac{1}{e^3}\right)-\ln 2$ 16.) $2\ln\left(\sqrt{x}+1\right)+C$ 17.) $\frac{\pi}{4}+\frac{\ln 2}{2}$ 18.) $\frac{1}{6}-\frac{1}{6e^3}$

19.) $\frac{4}{3}\left(1+\sqrt{x}\right)\sqrt{1+\sqrt{x}}+C$ 20.) $-2\cos\sqrt{x}+C$ 21.) $\frac{1}{2}\ln\left(x^2+2\right)+\frac{1}{x^2+2}+C$ 22.) $\frac{1}{3}\tan^{-1}\frac{1}{3}x+C$

23.) $\frac{1}{ab}\tan^{-1}\left(\frac{a}{b}x\right)+C$ 24.) $\frac{1}{2}\left(\sin^{-1}x\right)^2+C$ 25.) $\frac{\sqrt{3}}{4}$ 26.) $6\left(\sin x+1\right)^7-7\left(\sin x+1\right)^6+C$

27.) $2\ln 2$ 28.) $\ln\left|\sec x\right|+C$ 29.) $\ln\left|\sec x+\tan x\right|+C$ 30.) $\frac{1}{3}\sin^{-1}\left(x^3-1\right)+C$

31.) $\tan^{-1}\left(e^x\right)+C$ 32.) $\sin^{-1}\left(\frac{x}{4}\right)+C$ 33.) $\frac{1}{8\ln^2(\ln 4)}-\frac{1}{8\ln^2(\ln 5)}$ 34.) $\frac{9}{64}$

35.) $\frac{1}{2}x-\frac{1}{4}\sin 2x+C$ 36.) $\frac{\pi}{2}$ 37.) πr^2

Sample Problems - Solutions

Compute each of the following integrals. Please note that $\arcsin x$ is the same as $\sin^{-1} x$ and $\arctan x$ is the same as $\tan^{-1} x$.

1. $\displaystyle\int e^{-4x}\,dx$

   Solution: Let $u = -4x$. Then $du = -4dx$ and so $dx = -\frac{1}{4}du$. We now substitute in the integral

$$\int e^{-4x}\,dx = \int e^{u}\ \ -\frac{1}{4}du = -\frac{1}{4}\int e^{u}du = -\frac{1}{4}e^{u} + C = -\frac{1}{4}e^{-4x} + C$$

2. $\displaystyle\int_0^8 e^{-4x}\,dx$

   Solution:

$$\int_0^8 e^{-4x}\,dx = -\frac{1}{4}e^{-4x}\Big|_0^8 = -\frac{1}{4}\left(e^{-4(8)} - e^{-4(0)}\right) = -\frac{1}{4}\left(e^{-32} - e^{0}\right) = -\frac{1}{4}\left(e^{-32} - 1\right) = \frac{1}{4} - \frac{1}{4e^{32}}$$

3. $\displaystyle\int \left(x^2-2\right)\left(x^3-6x\right)^{207}\,dx$

   Solution: Let $u = x^3 - 6x$. Then $du = \left(3x^2-6\right)dx$ and so $dx = \frac{1}{(3x^2-6)}du$. We now substitute in the integral

$$\int \left(x^2-2\right)\left(x^3-6x\right)^{207}\,dx =$$

$$= \int \left(x^2-2\right)u^{207}\,\frac{1}{(3x^2-6)}du = \int \left(x^2-2\right)u^{207}\,\frac{1}{3\left(x^2-2\right)}du = \frac{1}{3}\int u^{207}\,du = \frac{1}{3}\left(\frac{u^{208}}{208}\right) + C$$

$$= \frac{1}{624}u^{208} + C = \frac{1}{624}\left(x^3-6x\right)^{208} + C$$

4. $\displaystyle\int e^{\cos x}\sin x\,dx$

   Solution: Let $u = \cos x$. Then $du = -\sin x dx$ and so $dx = -\frac{1}{\sin x}du$.

$$\int e^{\cos x}\sin x\,dx = \int e^{u}\sin x\,\frac{-1}{\sin x}du = -\int e^{u}du = -e^{u} + C = -e^{\cos x} + C$$

5. $\displaystyle\int_0^{\pi/2} \left(e^{\cos x}\sin x\right)\,dx$

   Solution:

$$\int_0^{\pi/2} \left(e^{\cos x}\sin x\right)\,dx = -e^{\cos x}\Big|_0^{\pi/2} = -\left(e^{\cos(\pi/2)} - e^{\cos 0}\right) = -\left(e^{0} - e^{1}\right) = e - 1$$

6. $\displaystyle\int \frac{3x}{(x^2+1)^7}\,dx$

Solution: Let $u = x^2+1$. Then $du = 2x dx$ and so $dx = \dfrac{1}{2x}du$.

$$\int \frac{3x}{(x^2+1)^7}\,dx = \int \frac{3x}{u^7}\,\frac{1}{2x}du = \frac{3}{2}\int \frac{1}{u^7}\,du = \frac{3}{2}\int u^{-7}\,du = \frac{3}{2}\frac{u^{-6}}{-6}+C = -\frac{1}{4u^6}+C = -\frac{1}{4(x^2+1)^6}+C$$

7. $\displaystyle\int \frac{12x^3}{3x^4+1}\,dx$

Solution: Let $u = 3x^4+1$. Then $du = 12x^3dx$ and so $dx = \dfrac{1}{12x^3}du$.

$$\int \frac{12x^3}{3x^4+1}\,dx = \int \frac{12x^3}{u}\,\frac{1}{12x^3}du = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left(3x^4+1\right)+C$$

8. $\displaystyle\int (-3x+4)\,e^{-3x^2+8x}\,dx$

Solution: Let $u = -3x^2+8x$. Then $du = (-6x+8)\,dx$ and so $dx = \dfrac{1}{(-6x+8)}du$.

$$\begin{aligned}\int (-3x+4)\,e^{-3x^2+8x}\,dx &= \int (-3x+4)\,e^u\,\frac{1}{(-6x+8)}du = \int (-3x+4)\,e^u\,\frac{1}{2(-3x+4)}du = \frac{1}{2}\int e^u\,du\\ &= \frac{1}{2}e^u + C = \frac{1}{2}e^{-3x^2+8x}+C\end{aligned}$$

9. $\displaystyle\int_0^2 (-3x+4)\,e^{-3x^2+8x}\,dx$

Solution:

$$\int_0^2 (-3x+4)\,e^{-3x^2+8x}\,dx = \frac{1}{2}e^{-3x^2+8x}\Big|_0^2 = \frac{1}{2}\left(e^{-3(2)^2+8(2)} - e^{-3(0)^2+8(0)}\right) = \frac{1}{2}\left(e^4-e^0\right) = \frac{1}{2}\left(e^4-1\right)$$

10. $\displaystyle\int \frac{x+5}{x^2+1}\,dx$

Solution:

$$\int \frac{x+5}{x^2+1}\,dx = \int\left(\frac{x}{x^2+1}+\frac{5}{x^2+1}\right)dx = \int \frac{x}{x^2+1}\,dx + \int \frac{5}{x^2+1}\,dx$$

These two integrals can be computed via very different methods. For the first integral, let $u = x^2+1$. Then $du = 2xdx$ and so $dx = \dfrac{1}{2x}du$.

$$\int \frac{x}{x^2+1}\,dx = \int \frac{x}{u}\,\frac{1}{2x}du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln\left(x^2+1\right)+C$$

and the second integral is

$$\int \frac{5}{x^2+1}\,dx = 5\int \frac{1}{x^2+1}\,dx = 5\tan^{-1}x + C$$

and so

$$\begin{aligned}\int \frac{x+5}{x^2+1}\,dx &= \int \frac{x}{x^2+1}\,dx + \int \frac{5}{x^2+1}\,dx = \frac{1}{2}\ln\left(x^2+1\right)+C_1+5\tan^{-1}x + C_2\\ &= \frac{1}{2}\ln\left(x^2+1\right)+5\tan^{-1}x + C\end{aligned}$$

11. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

Solution: Let $u = e^x + 1$. Then $du = e^x dx$ and so $dx = \dfrac{1}{e^x}du$.

$$\int \frac{e^x}{e^x+1}\,dx = \int \frac{e^x}{u}\,\frac{1}{e^x}du = \int \frac{1}{u}du = \ln|u| + C = \ln\left(e^x+1\right) + C$$

12. $\displaystyle\int \cos x \sin^5 x\; dx$

Solution: Let $u = \sin x$. Then $du = \cos x dx$ and so $dx = \dfrac{1}{\cos x}du$.

$$\int \cos x \sin^5 x\; dx = \int \cos x\; u^5\;\frac{1}{\cos x}du = \int u^5\; du = \frac{1}{6}u^6 + C = \frac{1}{6}\sin^6 x + C$$

13. $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

Solution: Let $u = x+1$. Then $du = dx$ and $x = u - 1$

$$\begin{aligned}\int \frac{x}{\sqrt{x+1}}\,dx &= \int \frac{u-1}{\sqrt{u}}\,du = \int (u-1)\,u^{-1/2}\,du = \int uu^{-1/2} - u^{-1/2}\,du = \int u^{1/2} - u^{-1/2}\,du \\ &= \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C\end{aligned}$$

14. $\displaystyle\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx$

Solution: We worked out the indefinite integral in the previous problem.

$$\begin{aligned}\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx &= \left(\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2}\right)\Bigg|_0^{15} \\ &= \left(\frac{2}{3}(15+1)^{3/2} - 2(15+1)^{1/2}\right) - \left(\frac{2}{3}(0+1)^{3/2} - 2(0+1)^{1/2}\right) \\ &= \left(\frac{2}{3}\left(16^{3/2}\right) - 2\left(16^{1/2}\right)\right) - \left(\frac{2}{3}\left(1^{3/2}\right) - 2\left(1^{1/2}\right)\right) = \left(\frac{2}{3}(64) - 2(4)\right) - \left(\frac{2}{3} - 2\right) \\ &= \frac{128}{3} - 8 - \frac{2}{3} + 2 = \frac{126}{3} - 6 = 36\end{aligned}$$

15. $\displaystyle\int_0^{3} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

Solution: Let $u = e^x + e^{-x}$. Then $du = \left(e^x - e^{-x}\right)dx$ and so $dx = \dfrac{1}{\left(e^x - e^{-x}\right)}du$. Also, when $x = 0$, then $u = e^0 + e^{-0} = 2$ and when $x = 3$, then $u = e^3 + e^{-3}$

$$\int_0^{3} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int_2^{e^3+e^{-3}} \frac{e^x - e^{-x}}{u}\,\frac{1}{\left(e^x - e^{-x}\right)}du = \int_2^{e^3+e^{-3}} \frac{1}{u}\,du = \ln|u|\Bigg|_2^{e^3+e^{-3}} = \ln\left(e^3 + e^{-3}\right) - \ln 2$$

16. $\displaystyle\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\,dx$

Solution: Let $u=\sqrt{x}+1$. Then $du=\dfrac{1}{2\sqrt{x}}dx$ and so $dx=2\sqrt{x}du$.

$$\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\,dx=\int \frac{1}{\sqrt{x}\,u}\,2\sqrt{x}du=2\int \frac{1}{u}\,du=2\ln|u|+C=2\ln\left(\sqrt{x}+1\right)+C$$

17. $\displaystyle\int_0^1 \frac{x+1}{x^2+1}\,dx$

Solution: We will first work out the indefinite integral. (For more details, pease refer to problem #10.)

$$\int \frac{x+1}{x^2+1}\,dx=\int \frac{x}{x^2+1}\,dx+\int \frac{1}{x^2+1}\,dx=\frac{1}{2}\ln\left(x^2+1\right)+\tan^{-1}x+C$$

and so the definite integral is.

$$\begin{aligned}\int_0^1 \frac{x+1}{x^2+1}\,dx &= \left(\frac{1}{2}\ln\left(x^2+1\right)+\tan^{-1}x\right)\Bigg|_0^1=\left(\frac{1}{2}\ln\left(1^2+1\right)+\tan^{-1}1\right)-\left(\frac{1}{2}\ln\left(0^2+1\right)+\tan^{-1}0\right)\\ &= \left(\frac{1}{2}\ln 2+\frac{\pi}{4}\right)-0=\frac{1}{2}\ln 2+\frac{\pi}{4}\end{aligned}$$

18. $\displaystyle\int_0^1 xe^{-3x^2}\,dx$

Solution: We will first work out the indefinite integral. Let $u=-3x^2$. Then $du=-6xdx$ and so $dx=-\dfrac{1}{6x}du$. We will also have to substitute the limits of the integration: when $x=0$, then $u=0$ and when $x=1$, then $u=-3$.

$$\int_0^1 xe^{-3x^2}\,dx=\int_0^{-3} xe^u\left(-\frac{1}{6x}du\right)=-\frac{1}{6}\int_0^{-3} e^u\,du=\frac{1}{6}\int_{-3}^0 e^u\,du=\frac{1}{6}e^u\Big|_{-3}^0=\frac{1}{6}\left(e^0-e^{-3}\right)=\frac{1}{6}\left(1-\frac{1}{e^3}\right)$$

19. $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx=\frac{4}{3}\left(1+\sqrt{x}\right)\sqrt{1+\sqrt{x}}+C$

Solution: We will use substitution. Let $u=1+\sqrt{x}$ then $du=\dfrac{1}{2\sqrt{x}}dx$ and so $dx=2\sqrt{x}du$

$$\begin{aligned}\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx &= \int \frac{\sqrt{u}}{\sqrt{x}}2\sqrt{x}du=2\int\sqrt{u}du=2\left(\frac{2}{3}\right)u^{3/2}+C=\frac{4}{3}\left(1+\sqrt{x}\right)^{3/2}+C\\ &= \frac{4}{3}\left(1+\sqrt{x}\right)\sqrt{1+\sqrt{x}}+C=\frac{4}{3}\sqrt{1+\sqrt{x}}+\frac{4}{3}\sqrt{x}\sqrt{1+\sqrt{x}}+C\end{aligned}$$

20. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Solution: Let $u=\sqrt{x}$. Then $du=\dfrac{1}{2\sqrt{x}}dx$ and so $dx=2\sqrt{x}du$.

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx=\int \frac{\sin u}{\sqrt{x}}\,2\sqrt{x}du=2\int \sin u\,du=-2\cos u+C=-2\cos\sqrt{x}+C$$

21. $\displaystyle\int \frac{x^3}{(x^2+2)^2}\,dx$

Solution: This integral is an interesting case of substitution. Let $u = x^2+2$. Then $du = 2xdx$ and so $dx = \dfrac{du}{2x}$ and also, $x^2 = u-2$.

$$\int \frac{x^3}{(x^2+2)^2}\,dx = \int \frac{x^3}{u^2}\,\frac{du}{2x} = \frac{1}{2}\int \frac{x^2}{u^2}\,du$$

It looks like we are 'stuck' with both x and u in the integrand. However, we can get rid of x^2 because it is $u-2$.

$$\frac{1}{2}\int \frac{x^2}{u^2}\,du = \frac{1}{2}\int \frac{u-2}{u^2}\,du = \frac{1}{2}\int \frac{1}{u} - \frac{2}{u^2}\,du = \frac{1}{2}\ln|u| + \frac{1}{u} + C = \frac{1}{2}\ln\left(x^2+2\right) + \frac{1}{x^2+2} + C$$

Note that because x^2+2 is positive for all values of x, the expression $\ln\left|x^2+2\right|$ could be simplified to $\ln\left(x^2+2\right)$.

22. $\displaystyle\int \frac{1}{x^2+9}\,dx$

Solution: The idea here is that we will 'get rid' of the 9 by factoring it out. If we used a substitution that would turn x^2 into $9u^2$, then the integrand would be $\dfrac{1}{9u^2+9} = \dfrac{1}{9}\cdot\dfrac{1}{u^2+1}$. We will pursue this substitution: $x^2 = 9u^2 \implies x = \pm 3u$. We select one of these.

Let $u = \dfrac{1}{3}x$. Then $du = \dfrac{1}{3}dx$ and so $dx = 3du$.

$$\int \frac{1}{x^2+9}\,dx = \int \frac{1}{9u^2+9}\,3du = \frac{3}{9}\int \frac{1}{u^2+9}\,du = \frac{1}{3}\arctan u + C = \frac{1}{3}\arctan\frac{x}{3} + C$$

23. $\displaystyle\int \frac{1}{a^2x^2+b^2}\,dx$

Solution: We need a substitution under which $a^2x^2 = b^2u^2$. This would be convenient because then

$$\frac{1}{a^2x^2+b^2} = \frac{1}{b^2u^2+b^2} = \frac{1}{b^2}\cdot\frac{1}{u^2+1}$$

So we will pursue this substitution. We solve $a^2x^2 = b^2u^2$ for a possible value of u and obtain $u = \dfrac{a}{b}x$. Then $du = \dfrac{a}{b}dx$ and so $\dfrac{b}{a}du = dx$.

$$\begin{aligned}\int \frac{1}{a^2x^2+b^2}\,dx &= \int \frac{1}{b^2u^2+b^2}\left(\frac{b}{a}du\right) = \int \frac{1}{b^2}\cdot\frac{1}{u^2+1}\cdot\frac{b}{a}\,du = \frac{b}{ab^2}\int\frac{1}{u^2+1}\,du = \frac{1}{ab}\tan^{-1}u + C \\ &= \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}x\right) + C\end{aligned}$$

24. $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Solution: Let $u = \sin^{-1}x$. We differentiate both sides and obtain that $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$ and so $dx = \sqrt{1-x^2}\,du$

$$\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int \frac{u}{\sqrt{1-x^2}}\,\sqrt{1-x^2}du = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}\left(\sin^{-1}x\right)^2 + C$$

25. $\int_0^{\pi/3} \sin 2x \sin x \, dx$

Solution:

$$\int_0^{\pi/3} \sin 2x \sin x \, dx = \int_0^{\pi/3} (2\sin x \cos x) \sin x \, dx = 2\int_0^{\pi/3} \sin^2 x \cos x \, dx$$

From here we will use substitution. Let $u = \sin x$ then $du = \cos x dx$. We will also need to compute the limits of the integral using u instead of x. When $x = 0$, then $u = \sin 0 = 0$ and when $x = \frac{\pi}{3}$, then $u = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$$2\int_0^{\pi/3} \sin^2 x \cos x \, dx = 2\int_0^{\sqrt{3}/2} u^2 \, du = \frac{2}{3} u^3\Big|_0^{\sqrt{3}/2} = \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{2}{3}\cdot\frac{3}{4}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$

26. $\int 42\cos x \sin x (\sin x + 1)^5 \, dx$

Solution: We will use substitution. Let $u = \sin x + 1$. then $du = \cos x dx$ and so $\frac{du}{\cos x} = dx$. Also, $u - 1 = \sin x$.

$$\begin{aligned}\int 42\cos x \sin x (\sin x + 1)^5 \, dx &= \int 42\cos x (u-1) u^5 \frac{du}{\cos x} = 42\int (u-1) u^5 \, du = 42\int u^6 - u^5 \, du \\ &= 42\left(\frac{u^7}{7} - \frac{u^6}{6}\right) + C = 6u^7 - 7u^6 + C = 6(\sin x + 1)^7 - 7(\sin x + 1)^6 + C\end{aligned}$$

27. $\int_e^{e^2} \frac{1}{x \ln \sqrt{x}} \, dx$

Solution:

$$\int_e^{e^2} \frac{1}{x\ln\sqrt{x}} \, dx = \int_e^{e^2} \frac{1}{x\left(\frac{1}{2}\right)\ln x} \, dx = \int_e^{e^2} \frac{2}{x\ln x} \, dx = 2\int_e^{e^2} \frac{1}{x\ln x} \, dx$$

Let $u = \ln x$. Then $du = \frac{1}{x}dx$. We will also have to substitute the limits of the integration: when $x = e$, then $u = \ln e = 1$ and when $x = e^2$, then $u = \ln e^2 = 2$.

$$2\int_e^{e^2} \frac{1}{x\ln x} \, dx = 2\int_e^{e^2} \frac{1}{\ln x}\frac{1}{x} \, dx = 2\int_1^2 \frac{1}{u} \, du = 2\int_1^2 \frac{1}{u} \, du = 2\ln|u|\Big|_1^2 = 2(\ln 2 - \ln 1) = 2\ln 2$$

28. $\int \tan x \, dx$

Solution: Let $u = \cos x$, then $du = -\sin x \, dx$ and so $dx = -\frac{du}{\sin x}$

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = \int \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right) = -\int \frac{1}{u} \, du = -\ln|u| + C = -\ln|\cos x| + C \\ &= \ln\left|(\cos x)^{-1}\right| + C = \ln|\sec x| + C\end{aligned}$$

29. $\displaystyle\int \sec x \; dx$

Solution: $\displaystyle\int \sec x \; dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \; dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx$

From here we will use substitution. Let $u = \sec x + \tan x$. Then $du = \left(\sec x \tan x + \sec^2 x\right) dx$ and so $dx = \dfrac{du}{\sec^2 x + \sec x \tan x}$

$$\begin{aligned}\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx &= \int \frac{\sec^2 x + \sec x \tan x}{u} \left(\frac{du}{\sec^2 x + \sec x \tan x}\right) = \int \frac{1}{u} \; du = \ln|u| + C \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

30. $\displaystyle\int \frac{x^2}{\sqrt{1-(x^3-1)^2}} \; dx$

Solution: Let $u = x^3 - 1$. Then $du = 3x^2 dx$ and so $dx = \dfrac{du}{3x^2}$

$$\int \frac{x^2}{\sqrt{1-(x^3-1)^2}} \; dx = \int \frac{x^2}{\sqrt{1-u^2}} \left(\frac{du}{3x^2}\right) = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}} \; du = \frac{1}{3}\arcsin u + C = \frac{1}{3}\arcsin\left(x^3-1\right) + C$$

31. $\displaystyle\int \frac{e^x}{e^{2x}+1} \; dx$

Solution: Let $u = e^x$. Then $du = e^x dx$.

$$\int \frac{e^x}{e^{2x}+1} \; dx = \int \frac{1}{u^2+1} \; e^x dx = \int \frac{1}{u^2+1} \; du = \arctan u + C = \arctan\left(e^x\right) + C$$

32. $\displaystyle\int \frac{1}{\sqrt{16-x^2}} \; dx$

Solution: The basic integral we see here is $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \; dx = \sin^{-1} x + C$ We will use substitution to get rid of the 16 in the denominator. Let use substitute u so that $x^2 = 16u^2$. Then $x = 4u$ and $dx = 4du$

$$\int \frac{1}{\sqrt{16-x^2}} \; dx = \int \frac{1}{\sqrt{16-16u^2}} \; (4du) = \int \frac{4}{\sqrt{16}\sqrt{1-u^2}} \; du = \int \frac{1}{\sqrt{1-u^2}} \; du = \sin^{-1} u + C = \sin^{-1}\left(\frac{x}{4}\right) + C$$

33. $\displaystyle\int_4^5 \frac{1}{4x \ln x \left[\ln(\ln x)\right]^3} dx$

Solution: Note that $\ln^3 \ln x$ is short for $\left[\ln(\ln x)\right]^3$. We will substitute $u = \ln(\ln x)$. Then $du = \dfrac{1}{\ln x} \cdot \dfrac{1}{x} dx$. We also substitute the limits of the integral: if $x = 4$, then $u = \ln(\ln 4)$ and if $x = 5$, then $u = \ln(\ln 5)$. Then

$$\begin{aligned} I &= \int_4^5 \frac{1}{4x \ln x \left[\ln(\ln x)\right]^3} dx = \int_4^5 \frac{1}{4\left[\ln(\ln x)\right]^3} \left(\frac{1}{x \ln x} dx\right) = \int_{\ln\ln 4}^{\ln\ln 5} \frac{1}{4u^3} du = \frac{1}{4} \int_{\ln\ln 4}^{\ln\ln 5} u^{-3} du \\ &= \frac{1}{4} \frac{u^{-2}}{-2}\Bigg|_{\ln\ln 4}^{\ln\ln 5} = -\frac{1}{8}\left(\frac{1}{(\ln\ln 5)^2} - \frac{1}{(\ln\ln 4)^2}\right) = \frac{1}{8\ln^2(\ln 4)} - \frac{1}{8\ln^2(\ln 5)}\end{aligned}$$

34. $\displaystyle\int_0^{\pi/3} \sin x\left(\cos x - \cos^3 x\right)\ dx$

Solution: We first simplify the integrand.

$$\int \sin x\left(\cos x - \cos^3 x\right)\ dx = \int \sin x \cos x\left(1 - \cos^2 x\right)\ dx = \int \sin x \cos x \sin^2 x\ dx = \int \sin^3 x \cos x\ dx$$

Let $u = \sin x$. Then $du = \cos x dx$. We will also need to compute the limits of the integral using u instead of x. When $x = 0$, then $u = \sin 0 = 0$ and when $x = \frac{\pi}{3}$, then $u = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$$\int_0^{\pi/3} \sin^3 x \cos x\ dx = \int_0^{\sqrt{3}/2} u^3\ \cos x dx = \int_0^{\sqrt{3}/2} u^3\ du = \left.\frac{u^4}{4}\right|_0^{\sqrt{3}/2} = \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)^4 = \frac{1}{4}\cdot\frac{9}{16} = \frac{9}{64}$$

35. $\displaystyle\int \sin^2 x\ dx$

Solution: Recall the double-angle formula for cosine and solve it for $\sin^2 x$.

$$\cos 2x = 1 - 2\sin^2 x \qquad \Longrightarrow \qquad \sin^2 x = \frac{1}{2}\left(1 - \cos 2x\right)$$

$$\int \sin^2 x\ dx = \int \frac{1}{2}\left(1 - \cos 2x\right)\ dx = \frac{1}{2}\int 1 - \cos 2x\ dx = \frac{1}{2}\left(\int 1\ dx - \int \cos 2x\ dx\right)$$

For the second integral, we substitute $u = 2x$ and $du = 2dx$.

$$\int \cos 2x\ dx = \int \cos u\ \frac{du}{2} = \frac{1}{2}\int \cos u\ du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$$

Thus the entire integral is

$$\begin{aligned}\int \sin^2 x\ dx &= \frac{1}{2}\left(\int 1\ dx - \int \cos 2x\ dx\right) = \frac{1}{2}\left(x + C_1 - \left(\frac{1}{2}\sin 2x + C_2\right)\right) = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x + C_3\right) \\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C\end{aligned}$$

36. $\displaystyle\int_{-1}^{1} \sqrt{1 - x^2}\ dx$

Solution: Let us first notice that the function is even and the limits of the integral are opposites.

$$\int_{-1}^{1} \sqrt{1 - x^2}\ dx = 2\int_0^1 \sqrt{1 - x^2}\ dx$$

This will make the computation somewhat easier. The following method is called trigonometric substitution. Let θ be so that $x = \cos\theta$. Then $\theta = \cos^{-1} x$. The substitution $\theta = \cos^{-1} x$. means that $x = \cos\theta$ and that θ is in the interval $[0, \pi]$. This is important to know because on $[0, \pi]$ $\sin\theta$ is non-negative and so $|\sin\theta| = \sin\theta$. We differentiate the statement $x = \cos\theta$ and obtain $dx = -\sin\theta d\theta$. We will also need to substitute the limits of integration. When $x = 0$, then $\theta = \cos^{-1} 0 = \frac{\pi}{2}$ and when $x = 1$, then $\theta = \cos^{-1} 1 = 0$.

$$2\int_0^1 \sqrt{1-x^2}\ dx = 2\int_{\pi/2}^0 \sqrt{1-\cos^2\theta}\ (-\sin\theta)\ d\theta = -2\int_{\pi/2}^0 \sqrt{\sin^2\theta}\ \sin\theta\ d\theta = -2\int_{\pi/2}^0 |\sin\theta|\ \sin\theta\ d\theta$$
$$= -2\int_{\pi/2}^0 \sin^2\theta\ d\theta = 2\int_0^{\pi/2} \sin^2\theta\ d\theta$$

In the previous problem we have computed the indefinite integral:

$$\int \sin^2 u\ du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$$

Thus the definite integral is

$$2\int_0^{\pi/2} \sin^2 u\ du = 2\left(\frac{1}{2}u - \frac{1}{4}\sin 2u\right)\Big|_0^{\pi/2} = \left(u - \frac{1}{2}\sin 2u\right)\Big|_0^{\pi/2} = \left(\frac{\pi}{2} - \frac{1}{2}\sin 2\left(\frac{\pi}{2}\right)\right) - \left(0 - \frac{1}{2}\sin 2(0)\right)$$
$$= \left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(0 - \frac{1}{2}\sin 0\right) = \left(\frac{\pi}{2} - \frac{1}{2}(0)\right) - 0 = \frac{\pi}{2}$$

Thus

$$\int_{-1}^1 \sqrt{1-x^2}\ dx = \frac{\pi}{2} \qquad \Longrightarrow \qquad 2\int_{-1}^1 \sqrt{1-x^2}\ dx = \pi$$

This is one of the definitions of π in modern mathematics. Also notice that we have just proved the area formula for the unit circle.

37. $2\int_{-r}^{r} \sqrt{r^2-x^2}\ dx$

Solution: First we notice that this function is even and the limits of integration are opposites:

$$2\int_{-r}^{r} \sqrt{r^2-x^2}\ dx = 4\int_0^r \sqrt{r^2-x^2}\ dx$$

We will apply a trigonometric substitution under which $x^2 = r^2\cos^2\theta$. This will be useful because $\sqrt{r^2-x^2}$ will become

$$\sqrt{r^2-x^2} = \sqrt{r^2-r^2\cos^2\theta} = \sqrt{r^2\left(1-\cos^2\theta\right)} = \sqrt{r^2}\sqrt{\sin^2\theta} = r\,|\sin\theta|$$

So, we solve for a possible value of θ in $x^2 = r^2\cos^2\theta$.

$$\begin{aligned} x^2 &= r^2\cos^2\theta \\ x &= r\cos\theta \\ \frac{x}{r} &= \cos\theta \\ \cos^{-1}\left(\frac{x}{r}\right) &= \theta \end{aligned}$$

So our substitution is $\theta = \cos^{-1}\left(\frac{x}{r}\right)$. That is the same as $\cos\theta = \frac{x}{r}$ and θ is in the interval $[0,\pi]$ (the range of the function $\cos^{-1}$). This is useful to know because on the interval $[0,\pi]$, sine is always non-negative and so $|\sin\theta|$ can be simplified as $\sin\theta$. We also differentiate both sides of the equation $\cos\theta = \frac{x}{r}$ and obtain that $-\sin\theta d\theta = \frac{dx}{r}$ and so $dx = -r\sin\theta d\theta$. We will need to substitute the limits as well. When $x=0$, then $\theta = \cos^{-1}\left(\frac{0}{r}\right) = \cos^{-1}0 = \frac{\pi}{2}$ and when $x=r$, then $\theta = \cos^{-1}\left(\frac{r}{r}\right) = \cos^{-1}1 = 0$. We are now ready to integrate:

$$
\begin{aligned}
I &= 4\int_0^r \sqrt{r^2-x^2}\,dx = 4\int_{\pi/2}^0 \sqrt{r^2-r^2\cos^2\theta}\,\left(-r\sin\theta d\theta\right) = 4\int_{\pi/2}^0 -r\sin\theta\sqrt{r^2\left(1-\cos^2\theta\right)}\,d\theta = \\
&= 4\int_{\pi/2}^0 -r\sin\theta r\sqrt{\sin^2\theta}\,d\theta = 4\int_{\pi/2}^0 -r^2\sin\theta\,|\sin\theta|\,d\theta = -4r^2\int_{\pi/2}^0 \sin\theta\cdot\sin\theta\,d\theta \\
&= -4r^2\int_{\pi/2}^0 \sin^2\theta\,d\theta = 4r^2\int_0^{\pi/2}\sin^2\theta\,d\theta
\end{aligned}
$$

We have already computed the antiderivative of $\sin^2\theta$ in problem 35:

$$\int \sin^2\theta\,d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$
\begin{aligned}
I &= 4r^2\int_0^{\pi/2}\sin^2\theta\,d\theta = 4r^2\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\Bigg|_0^{\pi/2} = r^2\left(2\theta-\sin 2\theta\right)\Big|_0^{\pi/2} \\
&= r^2\left[\left(2\left(\frac{\pi}{2}\right)-\sin 2\left(\frac{\pi}{2}\right)\right)-\left(2\left(0\right)-\sin 2\left(0\right)\right)\right] = r^2\left[\left(\pi-\sin\pi\right)-\left(0-\sin 0\right)\right] = r^2\left[\left(\pi-0\right)-0\right] = \pi r^2
\end{aligned}
$$

