

Claim: $\lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \cot^2 x \right) = \frac{2}{3}$

proof:

$$\lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x} \right) = \lim_{x \rightarrow 0} \left(\frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x} \right) = L$$

We apply l'Hôpital's rule (1):

$$\begin{aligned} L &= \lim_{x \rightarrow 0} \frac{2 \sin x \cos x - (2x \cos^2 x + x^2 2 \cos x (-\sin x))}{2x \sin^2 x + x^2 2 \sin x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin x \cos x - 2x \cos^2 x + 2x^2 \sin x \cos x}{2x \sin^2 x + 2x^2 \sin x \cos x} \end{aligned}$$

Recall that $2 \sin x \cos x = \sin 2x$ and $\cos^2 x = \frac{1 + \cos 2x}{2}$ and $\sin^2 x = \frac{1 - \cos 2x}{2}$ and so $2 \cos^2 x = 1 + \cos 2x$ and $2 \sin^2 x = 1 - \cos 2x$

$$\begin{aligned} L &= \lim_{x \rightarrow 0} \frac{2 \sin x \cos x - 2x \cos^2 x + 2x^2 \sin x \cos x}{2x \sin^2 x + 2x^2 \sin x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin 2x - x(1 + \cos 2x) + x^2 \sin 2x}{x(1 - \cos 2x) + x^2 \sin 2x} \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin 2x - x - x \cos 2x + x^2 \sin 2x}{x - x \cos 2x + x^2 \sin 2x} \right) && \text{the right side of the numerator almost matches} \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin 2x - 2x + x - x \cos 2x + x^2 \sin 2x}{x - x \cos 2x + x^2 \sin 2x} \right) && \text{the denominator} \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin 2x - 2x}{x - x \cos 2x + x^2 \sin 2x} + \frac{x - x \cos 2x + x^2 \sin 2x}{x - x \cos 2x + x^2 \sin 2x} \right) = \lim_{x \rightarrow 0} \left(\frac{\sin 2x - 2x}{x - x \cos 2x + x^2 \sin 2x} + 1 \right) \\ &= 1 + \lim_{x \rightarrow 0} \frac{\sin 2x - 2x}{x - x \cos 2x + x^2 \sin 2x} \end{aligned}$$

We apply l'Hôpital's rule again (2):

$$\begin{aligned} L &= 1 + \lim_{x \rightarrow 0} \frac{\sin 2x - 2x}{x - x \cos 2x + x^2 \sin 2x} \\ &= 1 + \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2}{1 - (\cos 2x - 2x \sin 2x) + (2x \sin 2x + 2x^2 \cos 2x)} \\ &= 1 + \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2}{1 - \cos 2x + 2x \sin 2x + 2x \sin 2x + 2x^2 \cos 2x} = 1 + \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2}{1 - \cos 2x + 4x \sin 2x + 2x^2 \cos 2x} \end{aligned}$$

We apply l'Hôpital's rule again (3):

$$\begin{aligned} L &= 1 + \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2}{1 - \cos 2x + 4x \sin 2x + 2x^2 \cos 2x} \\ &= 1 + \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2}{2 \sin 2x + (4 \sin 2x + 8x \cos 2x) + (4x \cos 2x - 4x^2 \sin 2x)} \\ &= 1 + \lim_{x \rightarrow 0} \frac{-4 \sin 2x}{2 \sin 2x + 4 \sin 2x + 8x \cos 2x + 4x \cos 2x - 4x^2 \sin 2x} \\ &= 1 + \lim_{x \rightarrow 0} \frac{-4 \sin 2x}{2 \sin 2x + 4 \sin 2x + 12x \cos 2x - 4x^2 \sin 2x} \\ &= 1 + \lim_{x \rightarrow 0} \frac{-4 \sin 2x}{6 \sin 2x + 12x \cos 2x - 4x^2 \sin 2x} = 1 + \lim_{x \rightarrow 0} \frac{-2 \sin 2x}{3 \sin 2x + 6x \cos 2x - 2x^2 \sin 2x} \end{aligned}$$

We apply l'Hôpital's rule again (4):

$$\begin{aligned} L &= 1 + \lim_{x \rightarrow 0} \frac{-2 \sin 2x}{3 \sin 2x + 6x \cos 2x - 2x^2 \sin 2x} \\ &= 1 + \lim_{x \rightarrow 0} \frac{-4 \cos 2x}{6 \cos 2x + (6 \cos 2x - 12x \sin 2x) - (4x \sin 2x + 4x^2 \cos 2x)} \\ &= 1 + \lim_{x \rightarrow 0} \frac{-4 \cos 2x}{6 \cos 2x + 6 \cos 2x - 12x \sin 2x - 4x \sin 2x - 4x^2 \cos 2x} \\ &= 1 + \lim_{x \rightarrow 0} \frac{-4 \cos 2x}{12 \cos 2x - 16x \sin 2x - 4x^2 \cos 2x} \\ &= 1 + \lim_{x \rightarrow 0} \frac{-\cos 2x}{3 \cos 2x - 4x \sin 2x - x^2 \cos 2x} = 1 + \frac{-1}{3} = \frac{2}{3} \end{aligned}$$