Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\dfrac{0}{0}$ or $\pm\dfrac{\infty}{\infty}$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

If the limit on the right side exists.

Sample Problems

1. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$
2. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x + \sin x}$
3. $\displaystyle\lim_{x\to\pi^-} \frac{\sin x}{1 - \cos x}$
4. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x + 2\sin x}$
5. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x}$
6. $\displaystyle\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x}$
7. $\displaystyle\lim_{x\to 0^+} x\ln x$
8. $\displaystyle\lim_{x\to 0} \frac{5^x - 1}{x^3}$
9. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x^2}$
10. $\displaystyle\lim_{x\to 1} \frac{\ln x}{x^2 - x}$
11. $\displaystyle\lim_{x\to 0} \frac{4x - \sin 4x}{x^3}$
12. $\displaystyle\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$
13. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x} - \ln x}{\sqrt[3]{x}}$
14. $\displaystyle\lim_{x\to 0} \frac{e^{x^2} + 10}{1 - \cos x}$

Practice Problems

1. $\displaystyle\lim_{x\to 0^+} \frac{e^x - 1}{x^2}$
2. $\displaystyle\lim_{x\to 0} \frac{e^{3x} - 1}{5x}$
3. $\displaystyle\lim_{a\to 1} \frac{3a^2 - 2a - 1}{5a^2 - a - 4}$
4. $\displaystyle\lim_{y\to\infty} \frac{\ln y}{\sqrt[3]{y}}$
5. $\displaystyle\lim_{x\to\pi} \frac{\sin x}{x - \pi}$
6. $\displaystyle\lim_{m\to 2} \frac{m^5 - 32}{m^3 - 8}$
7. $\displaystyle\lim_{\theta\to\pi/2} \frac{\tan\theta}{\tan 5\theta}$
8. $\displaystyle\lim_{x\to\infty} \frac{x}{\ln(x+1)}$
9. $\displaystyle\lim_{x\to 0} \frac{x^3}{\tan x - x}$
10. $\displaystyle\lim_{\beta\to 0} \frac{\sin\beta - \beta}{\tan\beta - \beta}$
11. $\displaystyle\lim_{x\to 0} \left(x^2 e^{1/x^2}\right)$
12. $\displaystyle\lim_{p\to 0} \frac{e^{3p} - 1}{\sin 2p}$
13. $\displaystyle\lim_{x\to 0} \frac{e^{(x^2)} + 10}{1 - \cos x}$
14. $\displaystyle\lim_{x\to 1} \frac{x^{2/3} - x^{1/2}}{x - 1}$
15. $\displaystyle\lim_{\alpha\to 0} \frac{\alpha}{\arctan 2\alpha}$

Sample Problems - Answers

1.) ∞ 2.) 0 3.) 0 4.) $\dfrac{1}{3}$ 5.) ∞ 6.) 6 7.) 0 8.) ∞ 9.) $-\dfrac{1}{2}$ 10.) 1

11.) $\dfrac{32}{3}$ 12.) $-\dfrac{1}{2}$ 13.) ∞ 14.) ∞

Practice Problems - Answers

1.) ∞ 2.) $\dfrac{3}{5}$ 3.) $\dfrac{4}{9}$ 4.) 0 5.) -1 6.) $\dfrac{20}{3}$ 7.) 5 8.) ∞ 9.) 3 10.) $-\dfrac{1}{2}$ 11.) ∞

12.) $\dfrac{3}{2}$ 13.) ∞ This is NOT an indeterminate! 14.) $\dfrac{1}{6}$ 15.) $\dfrac{1}{2}$

Sample Problems - Solutions

1. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$

   Solution: This is an $\dfrac{\infty}{\infty}$ type of an indeterminate, so we can apply l'Hôpital's rule. We differentiate both numerator and denominator:

   $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x}$$

   $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$ is an $\dfrac{\infty}{\infty}$ type of an indeterminate, so we can apply l'Hôpital's rule again

   $$\lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \boxed{\infty}$$

   $\displaystyle\lim_{x\to\infty} \frac{e^x}{2}$ is no longer an indeterminate because the numerator approaches infinity while the denominator approaches 2. This limit is ∞.

2. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x + \sin x}$

   Solution: $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x + \sin x}$ is a $\dfrac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

   $$\lim_{x\to 0} \frac{\cos x - 1}{x + \sin x} = \lim_{x\to 0} \frac{-\sin x}{1 + \cos x} = \frac{0}{2} = \boxed{0}$$

3. $\displaystyle\lim_{x\to \pi^-} \frac{\sin x}{1 - \cos x}$

   Solution: This limit does not qualify for l'Hôpital's rule because substituting π into the expression does NOT result in an indeterminate.

   $$\lim_{x\to \pi^-} \frac{\sin x}{1 - \cos x} = \frac{0}{1-(-1)} = \frac{0}{2} = \boxed{0}$$

   If applied the rule, we would get an incorrect answer, $-\infty$. **So, it is very important to check for the conditions of the rule.**

4. $\displaystyle\lim_{x\to 0}\frac{\sin x}{x+2\sin x}$

Solution: $\displaystyle\lim_{x\to 0}\frac{\sin x}{x+2\sin x}$ a $\dfrac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x\to 0}\frac{\sin x}{x+2\sin x}=\lim_{x\to 0}\frac{\cos x}{1+2\cos x}$$

$\displaystyle\lim_{x\to 0}\frac{\cos x}{1+2\cos x}$ is no longer an indeterminate: we get $\dfrac{1}{3}$ when we substitute $x=0$.

$$\lim_{x\to 0}\frac{\cos x}{1+2\cos x}=\frac{1}{1+2}=\boxed{\frac{1}{3}}$$

Note that we didn't really need L'Hôpital's rule to compute this limit. Recall that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1$ and then of course its reciproal approaches 1 as well: $\displaystyle\lim_{x\to 0}\frac{x}{\sin x}=1$. We can compute this limit by dividing both numerator and denominator by $\sin x$.

$$\lim_{x\to 0}\frac{\sin x}{x+2\sin x}=\lim_{x\to 0}\frac{1}{\dfrac{x}{\sin x}+2}=\frac{1}{\displaystyle\lim_{x\to 0}\frac{x}{\sin x}+2}=\frac{1}{1+2}=\frac{1}{3}$$

5. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}}{\ln x}$

Solution: $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}}{\ln x}$ is an $\dfrac{\infty}{\infty}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x\to\infty}\frac{\sqrt{x}}{\ln x}=\lim_{x\to\infty}\frac{\dfrac{1}{2\sqrt{x}}}{\dfrac{1}{x}}=\lim_{x\to\infty}\frac{1}{2\sqrt{x}}\frac{x}{1}=\lim_{x\to\infty}\frac{1}{2}\sqrt{x}=\boxed{\infty}$$

6. $\displaystyle\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x-\sin x}$

Solution: We substitute x into the expression and get a $\dfrac{0}{0}$ type of an indeterminate. So we can apply l'Hôpital's rule.

$$\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x-\sin x}=\lim_{x\to 0}\frac{2\cos x-\cos 2x\,(2)}{1-\cos x}=\lim_{x\to 0}\frac{2\,(\cos x-\cos 2x)}{1-\cos x}$$

This is still a $\dfrac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\lim_{x\to 0}\frac{2\,(\cos x-\cos 2x)}{1-\cos x}=\lim_{x\to 0}\frac{2\,(-\sin x+\sin 2x\,(2))}{\sin x}=\lim_{x\to 0}\frac{2\,(-\sin x+2\sin 2x)}{\sin x}$$

This is still a $\dfrac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\lim_{x\to 0}\frac{2\,(-\sin x+2\sin 2x)}{\sin x}=\lim_{x\to 0}\frac{2\,(-\cos x+2\cos 2x\,(2))}{\cos x}=\lim_{x\to 0}\frac{2\,(-\cos x+4\cos 2x)}{\cos x}$$

This is no longer an indeterminate:

$$\lim_{x\to 0}\frac{2\,(-\cos x+4\cos 2x)}{\cos x}=\frac{2\,(-1+4)}{1}=\boxed{6}$$

7. $\lim_{x\to 0^+} x\ln x$

Solution: Although it doesn't appear so, this is either a $\frac{0}{0}$ or an $\frac{\infty}{\infty}$ type of an indeterminate after we re-write it: $\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$ is an $\frac{-\infty}{\infty}$ type of an indeterminate and $\lim_{x\to 0^+}\frac{x}{\frac{1}{\ln x}}$ is a $\frac{0}{0}$ type of an indeterminate. We will use the first form because it seems to yield for slightly easier computation.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}\frac{1}{x}\left(-\frac{x^2}{1}\right) = \lim_{x\to 0^+}(-x) = \boxed{0}$$

8. $\lim_{x\to 0}\frac{5^x-1}{x^3}$

Solution: This is a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x\to 0}\frac{5^x-1}{x^3} = \lim_{x\to 0}\frac{(\ln 5)\,5^x}{3x^2} = \boxed{\infty}$$

In $\lim_{x\to 0}\frac{(\ln 5)\,5^x}{3x^2}$ the numerator approaches 1 while the denominator approaches zero and is positive. This limit is ∞.

9. $\lim_{x\to 0}\frac{\cos x-1}{x^2}$

Solution: This is a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x\to 0}\frac{\cos x-1}{x^2} = \lim_{x\to 0}\frac{-\sin x}{2x}$$

This is still a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\lim_{x\to 0}\frac{-\sin x}{2x} = \lim_{x\to 0}\frac{-\cos x}{2} = \boxed{-\frac{1}{2}}$$

10. $\lim_{x\to 1}\frac{\ln x}{x^2-x}$

Solution: This is a $\frac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x\to 1}\frac{\ln x}{x^2-x} = \lim_{x\to 1}\frac{\frac{1}{x}}{2x-1} = \frac{1}{2-1} = \boxed{1}$$

11. $\displaystyle\lim_{x\to 0}\frac{4x-\sin 4x}{x^3}$

Solution: This is a $\dfrac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

$$\lim_{x\to 0}\frac{4x-\sin 4x}{x^3}=\lim_{x\to 0}\frac{4-4\cos 4x}{3x^2}$$

This is still a $\dfrac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\lim_{x\to 0}\frac{4-4\cos 4x}{3x^2}=\lim_{x\to 0}\frac{16\sin 4x}{6x}=\lim_{x\to 0}\frac{8\sin 4x}{3x}$$

This is still a $\dfrac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\lim_{x\to 0}\frac{8\sin 4x}{3x}=\lim_{x\to 0}\frac{32\cos 4x}{3}=\boxed{\frac{32}{3}}$$

12. $\displaystyle\lim_{x\to 0}\frac{\sin x-\tan x}{x^3}=-\frac{1}{2}$

Solution: This is a $\dfrac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule.

Recall that $\dfrac{d}{dx}\tan x=\dfrac{1}{\cos^2 x}$

$$\lim_{x\to 0}\frac{\sin x-\tan x}{x^3}=\lim_{x\to 0}\frac{\cos x-\dfrac{1}{\cos^2 x}}{3x^2}=\lim_{x\to 0}\frac{\cos x-(\cos x)^{-2}}{3x^2}$$

This is still a $\dfrac{0}{0}$ type of an indeterminate, so we can apply l'Hôpital's rule again.

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x-(\cos x)^{-2}}{3x^2} &= \lim_{x\to 0}\frac{-\sin x-(-2)(\cos x)^{-3}(-\sin x)}{6x}=\lim_{x\to 0}\frac{-\sin x-2\sin x(\cos x)^{-3}}{6x}\\ &= \lim_{x\to 0}\frac{\sin x\left(-1-\dfrac{2}{\cos^3 x}\right)}{6x}=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\left(-1-\dfrac{2}{\cos^3 x}\right)}{6}=1\cdot\frac{-1-2}{6}=\boxed{-\frac{1}{2}}\end{aligned}$$

If we didn't notice the limit $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$ in the expressions, we can still get the answer by applying l'Hôpital's rule one more time.

$$\lim_{x\to 0}\frac{-\sin x-2\sin x(\cos x)^{-3}}{6x}=$$

$$\begin{aligned}&=\lim_{x\to 0}\frac{\sin x\left(-1-2(\cos x)^{-3}\right)}{6x}=\lim_{x\to 0}\frac{\cos x\left(-1-2(\cos x)^{-3}\right)+\sin x\left(-2(-3)(\cos x)^{-4}(-\sin x)\right)}{6}\\ &=\lim_{x\to 0}\frac{\cos x\left(-1-\dfrac{2}{\cos^3 x}\right)+\sin x\left(-6\dfrac{\sin x}{\cos^4 x}\right)}{6}=\frac{1\cdot\left(-1-\dfrac{2}{1}\right)+0\cdot\left(-6\dfrac{0}{1}\right)}{6}=-\frac{1}{2}\end{aligned}$$

13. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x}-\ln x}{\sqrt[3]{x}}=\lim\limits_{x\to\infty}\dfrac{\dfrac{1}{2\sqrt{x}}-\dfrac{1}{x}}{\dfrac{1}{3}x^{-2/3}}=\lim\limits_{x\to\infty}\dfrac{\dfrac{\sqrt{x}}{2}-1}{\dfrac{1}{3}x^{1/3}}=\lim\limits_{x\to\infty}\dfrac{\dfrac{1}{4\sqrt{x}}}{\dfrac{1}{9}x^{-2/3}}=\lim\limits_{x\to\infty}\dfrac{\dfrac{\sqrt{x}}{4}}{\dfrac{1}{9}x^{1/3}}=\infty$

Solution: In case of this problem, it will take some work to determine whether we can use L'Hôpital's rule for this limit. The numerator itself is an $\infty-\infty$ type of an indeterminate. We first factor out $\sqrt{x}$.

$$\lim_{x\to\infty}\left(\sqrt{x}-\ln x\right)=\lim_{x\to\infty}\sqrt{x}\left(1-\frac{\ln x}{\sqrt{x}}\right)$$

Inside the parentheses, $\dfrac{\ln x}{\sqrt{x}}$ is an $\dfrac{\infty}{\infty}$ type of an indeterminate. We apply l'Hôpital's rule:

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{\dfrac{1}{x}}{\dfrac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{1}{x}\frac{2\sqrt{x}}{1}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

This the numerator is

$$\lim_{x\to\infty}\left(\sqrt{x}-\ln x\right)=\lim_{x\to\infty}\sqrt{x}\left(1-\frac{\ln x}{\sqrt{x}}\right)=\lim_{x\to\infty}\sqrt{x}\lim_{x\to\infty}\left(1-\frac{\ln x}{\sqrt{x}}\right)=\left(\lim_{x\to\infty}\sqrt{x}\right)(1-0)=\infty$$

Thus the numerator approaches ∞, and so we have an $\dfrac{\infty}{\infty}$ type of an indeterminate. We apply l'Hôpital's rule.

$$\lim_{x\to\infty}\frac{\sqrt{x}-\ln x}{\sqrt[3]{x}}=\lim_{x\to\infty}\frac{\dfrac{1}{2\sqrt{x}}-\dfrac{1}{x}}{\dfrac{1}{3}x^{-2/3}}$$

We multiply both numerator and denominator by x :

$$\lim_{x\to\infty}\frac{\dfrac{1}{2\sqrt{x}}-\dfrac{1}{x}}{\dfrac{1}{3}x^{-2/3}}=\lim_{x\to\infty}\frac{\dfrac{\sqrt{x}}{2}-1}{\dfrac{1}{3}x^{1/3}}$$

This is still an $\dfrac{\infty}{\infty}$ type of an indeterminate, so we apply l'Hôpital's rule again.

$$\begin{aligned}\lim_{x\to\infty}\frac{\dfrac{\sqrt{x}}{2}-1}{\dfrac{1}{3}x^{1/3}}&=\lim_{x\to\infty}\frac{\dfrac{1}{4\sqrt{x}}}{\dfrac{1}{9}x^{-2/3}}=\lim_{x\to\infty}\frac{\dfrac{1}{4}x^{-1/2}}{\dfrac{1}{9}x^{-2/3}}=\lim_{x\to\infty}\frac{9x^{-1/2-(-2/3)}}{4}=\lim_{x\to\infty}\frac{9}{4}x^{\frac{2}{3}-\frac{1}{2}}=\lim_{x\to\infty}\frac{9}{4}x^{\frac{1}{6}}\\&=\lim_{x\to\infty}\frac{9}{4}\sqrt[6]{x}=\boxed{\infty}\end{aligned}$$

14. $\lim\limits_{x\to 0}\dfrac{e^{x^2}+10}{1-\cos x}$

This limit can not be solved using l'Hôpital's rule because it is not an indeterminate. The numerator approaches 11, the denominator approaches 0^+ and so the quotient approaches ∞. If we apply l'Hôpital's rule, we will get a wrong result:

$$\lim_{x\to 0}\frac{e^{x^2}+10}{1-\cos x}=\lim_{x\to 0}\frac{2xe^{x^2}}{\sin x}=\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}2e^{x^2}=1\cdot 2=2\qquad\text{incorrect}$$

Before applying l'Hôpital's rule, always check first whether the conditions for it hold.

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