Definition: $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$

Theorem: $\lim\limits_{n\to\infty}\left(1+\dfrac{a}{n}\right)^n = e^a$

Proof: If $a=0$, then clearly $\lim\limits_{n\to\infty}\left(1+\dfrac{a}{n}\right)^n = \lim\limits_{n\to\infty}\left(1+\dfrac{0}{n}\right)^n = \lim\limits_{n\to\infty} 1^n = 1$. If $a\neq 0$, then

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{a}}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{a}}\right)^{\frac{n}{a}\cdot a} = \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{n}{a}}\right)^{\frac{n}{a}}\right]^a$$

Define $m=\dfrac{n}{a}$. Since a is fixed, m approaches infinity (or negative infinity if a is negative) as n approaches infinity.

$$\left[\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{a}}\right)^{\frac{n}{a}}\right]^a = \left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^a = e^a$$

Practice Problems

Assume that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad\text{and}\quad \lim_{n\to-\infty}\left(1+\frac{1}{n}\right)^n = e$$

and compute each of the following limits.

1. $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n+2}$
2. $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{2n}$
3. $\lim\limits_{n\to\infty}\left(1+\dfrac{5}{n}\right)^{n}$
4. $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{n}$
5. $\lim\limits_{n\to\infty}\left(\dfrac{n}{n+2}\right)^{n}$

Answers - Practice Problems

1.) e 2.) e^2 3.) e^5 4.) $\dfrac{1}{e}$ 5.) $\dfrac{1}{e^2}$

Solutions - Practice Problems

Assume that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad \text{and} \quad \lim_{n\to-\infty}\left(1+\frac{1}{n}\right)^n = e$$

and compute each of the following limits.

1. $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+2}$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2} = e\cdot 1 = e$$

2. $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^2 = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^2 = e^2$$

3. $\displaystyle\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^{n}$

$$\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^{n} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{5}}\right)^{n} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{5}}\right)^{\frac{n}{5}\cdot 5} = \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{n}{5}}\right)^{\frac{n}{5}}\right]^5$$

Let $m=\dfrac{n}{5}$. As n approaches infinity, so does m.

$$\left[\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{5}}\right)^{\frac{n}{5}}\right]^5 = \left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^5 = e^5$$

4. $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^n$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{-1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{-1}}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{-1}}\right)^{\frac{n}{-1}\cdot(-1)}$$
$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{-n}\right)^{(-n)}\right]^{(-1)} = \left[\lim_{n\to\infty}\left(1+\frac{1}{-n}\right)^{(-n)}\right]^{(-1)}$$

Let $m=-n$

$$\left[\lim_{n\to\infty}\left(1+\frac{1}{-n}\right)^{(-n)}\right]^{(-1)} = \left[\lim_{m\to-\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{(-1)} = e^{-1} = \frac{1}{e}$$

5. $\lim\limits_{n\to\infty}\left(\dfrac{n}{n+2}\right)^n$

$$\lim_{n\to\infty}\left(\frac{n}{n+2}\right)^n = \lim_{n\to\infty}\left(\frac{1}{\frac{n+2}{n}}\right)^n = \lim_{n\to\infty}\left(\left(\frac{n+2}{n}\right)^{(-1)}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{2}{n}\right)^n\right]^{(-1)}$$
$$= \left[\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n\right]^{(-1)} = \left(e^2\right)^{(-1)} = e^{-2} = \frac{1}{e^2}$$

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