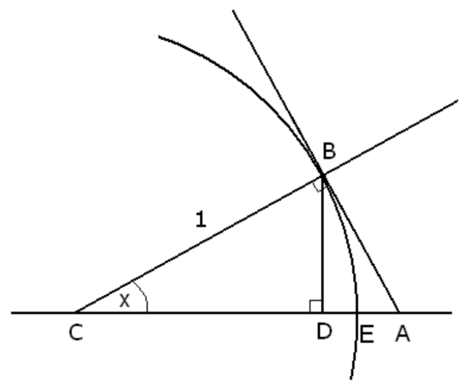
Theorem 1: $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

Proof: This theorem and the next one are necessary for differentiating $\sin x$ and $\cos x$. Recall a theorem: Let r be the radius of a circle. If α is measured in radians, then the area of a sector with a central angle of α is $A_{\text{sector}} = \dfrac{\alpha r^2}{2}$. (Notation: $\overline{AB}$ will denote the length of line segment AB.)

Let x be a very small positive angle, measured in radians, drawn into a unit circle as shown on the picture below. Let B be the point where the unit circle intersects the ray determined by x. We then draw a tangent line to the circle at point B. Let A be the point where the tangent line intersects the $x-$axis. We also draw a vertical line through B. Let D be the point where this vertical line intersects the $x-$axis. Finally, let us denote by E the point with coordinates $(0, 1)$



The proof will be based on the following fact: because they include each other, the following three areas can be easily compared:

$$\text{Area of triangle } CDB \ \leq \ \text{Area of sector } CEB \ \leq \ \text{Area of triangle } ABC$$

Area of triangle CDB: the horizontal side, $\overline{CD} = \cos x$ and the vertical side, $\overline{DB} = \sin x$. Since this is a right triangle, the area is: $A_{CDB} = \dfrac{1}{2} \sin x \cos x$

Area of sector CEB: $A_{\text{sector}} = \dfrac{1^2 x}{2} = \dfrac{x}{2}$

Area of triangle ABC: there is a right angle at point B because the tangent line drawn to a circle is perpendicular to the radius drawn to the point of tangency. So the area is $A_{ABC} = \dfrac{1}{2}\overline{AB} \cdot \overline{BC}$. Clearly $\overline{BC} = 1$. To compute $\overline{AB}$, in triangle ABC, $\tan x = \dfrac{\overline{AB}}{1}$ and so $\overline{AB} = \tan x$.

Area of triangle ABC: $\dfrac{1}{2}(1)(\tan x) = \dfrac{\tan x}{2}$ or $\dfrac{\sin x}{2\cos x}$. So now

$$\text{Area of triangle } CDB \ \leq \ \text{Area of sector } CEB \ \leq \ \text{Area of triangle } ABC$$

translates to

$$\frac{1}{2} \sin x \cos x \leq \frac{x}{2} \leq \frac{\sin x}{2\cos x}$$

Let us divide all three sides by $\dfrac{\sin x}{2}$. Because x is small and positive, $\dfrac{\sin x}{2}$ is positive and so we do not need to reverse the inequality signs.

$$\cos x \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

Suppose now that x approaches zero. Then both $\cos x$ and $\dfrac{1}{\cos x}$ approach 1. By the sandwich principle, $\dfrac{x}{\sin x}$, the quantity locked in between those two must also approach 1.

$$\begin{array}{ccccc} \cos x & \leq & \dfrac{x}{\sin x} & \leq & \dfrac{1}{\cos x} \\ \downarrow & & & & \downarrow \\ 1 & & & & 1 \end{array}$$

If $\dfrac{x}{\sin x}$ approaches 1, so does its reciprocal, $\dfrac{\sin x}{x}$.

So far, we have proven the statement for positive values of x, that is, $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{x} = 1$. A similar argument works for negative values of x.

$$\boxed{\text{Theorem 2: } \lim_{x\to 0} \frac{\cos x - 1}{x} = 0}$$

Proof:

$$\begin{aligned} \lim_{x\to 0} \frac{\cos x - 1}{x} &= \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot 1 = \lim_{x\to 0} \left(\frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} \right) = \lim_{x\to 0} \frac{\cos^2 x - 1}{x\left(\cos x + 1\right)} = \lim_{x\to 0} \frac{-\left(1 - \cos^2 x\right)}{x\left(\cos x + 1\right)} \\ &= \lim_{x\to 0} \frac{-\sin^2 x}{x\left(\cos x + 1\right)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{-\sin x}{\cos x + 1} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{-\sin x}{\cos x + 1} = 1 \cdot 0 = 0 \end{aligned}$$

Sample Problems

1. Recall that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$. Use this fact to compute each of the following limits.

   a) $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{x}$

   b) $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x}$

   c) $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x^2}$

   d) $\displaystyle\lim_{x\to 0} \frac{1 - \cos 3x}{x^2}$

   e) $\displaystyle\lim_{x\to 0} \frac{\tan x}{x}$

   f) $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{\sin 6x}$

   g) $\displaystyle\lim_{\theta\to \pi/4} \frac{\tan \theta - 1}{\theta - \dfrac{\pi}{4}}$

   g) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1 - \cos x}}{x}$

2. a) Find the perimeter of a $15-$sided reguar polygon written into a circle with radius 10m.

   b) Find the perimeter of an $n-$sided regular polygon written into a circle with radius R. Use radians to measure angles.

   c) Find the limit of the perimeter of an $n-$sided regular polygon written into a circle with radius R as n approaches infinity. Use radians to measure angles.

3. a) Find the area of a $15-$sided regular polygon written into a circle with radius 10m.

   b) Find the area of an $n-$sided regular polygon written into a circle with radius R. Use radians to measure angles.

   c) Find the limit of the area of an $n-$sided regular polygon written into a circle with radius R as n approaches infinity. Use radians to measure angles.

Practice Problems

Compute each of the following limits.

1. $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{x}$
2. $\displaystyle\lim_{x\to 0} \frac{2\sin x\cos x}{x}$
3. $\displaystyle\lim_{\theta\to 0} \frac{\cos 3\theta\sin 3\theta}{\theta}$
4. $\displaystyle\lim_{x\to 0} \frac{\sin 2x+\sin 4x}{x}$
5. $\displaystyle\lim_{\theta\to 0} \frac{\tan 4\theta}{5\theta}$
6. $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{\sin 3x}$
7. $\displaystyle\lim_{\theta\to 0} \frac{\theta}{\sin 3\theta}$
8. $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{3x^2}$
9. $\displaystyle\lim_{\theta\to 0} \frac{\sin\left(\theta^2\right)}{\theta\tan\theta}$
10. $\displaystyle\lim_{x\to 0} \frac{\tan 6x}{3x}$
11. $\displaystyle\lim_{x\to 0} \frac{\sin x\tan x}{x^2}$
12. $\displaystyle\lim_{x\to 0} \frac{\sin 2x\tan 3x}{x^2}$
13. $\displaystyle\lim_{\theta\to 0} \frac{\theta\sin 2\theta}{2-2\cos^2\theta}$
14. $\displaystyle\lim_{x\to 0} \frac{x}{\sin 2x}$
15. $\displaystyle\lim_{x\to 0} \frac{\tan x}{\tan 4x}$
16. $\displaystyle\lim_{x\to 0} \frac{6x}{\sin 4x+\sin 3x}$
17. $\displaystyle\lim_{\theta\to 0} \frac{2\theta^2}{1-\cos\theta}$
18. $\displaystyle\lim_{x\to\pi/2} \frac{\sin x-1}{x-\frac{\pi}{2}}$
19. $\displaystyle\lim_{\theta\to 0} \frac{\theta\sin\theta}{1-\cos\theta}$

Answers - Sample Problems

1. a) 5 b) 0 c) $\frac{1}{2}$ d) $\frac{9}{2}$ e) 1 f) $\frac{5}{6}$ g) 2 h) undefined
2. a) $300\sin\left(\frac{\pi}{15}\right)\text{m}\approx 62.3735\text{m}$ b) $2nR\sin\frac{\pi}{n}$ c) $2\pi R$
3. a) $750\sin\left(\frac{2\pi}{15}\right)\text{m}^2\approx 305.0525\text{m}^2$ b) $\frac{1}{2}nR^2\sin\frac{2\pi}{n}$ c) πR^2

Answers - Practice Problems

1. 5 2. 2 3. 3 4. 6 5. $\frac{4}{5}$ 6. $\frac{4}{3}$ 7. $\frac{1}{3}$ 8. $\frac{1}{3}$ 9. 1 10. 2

11. 1 12. 6 13. 1 14. $\frac{1}{2}$ 15. $\frac{1}{4}$ 16. $\frac{6}{7}$ 17. 4 18. 0 19. 2

Sample Problems - Solutions

1. Recall that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1$. Use this fact to compute each of the following limits.

a) $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{x}$

$$\lim_{x\to 0}\frac{\sin 5x}{x}=\lim_{x\to 0}\frac{\sin 5x}{x}\cdot 1=\lim_{x\to 0}\frac{\sin 5x}{x}\cdot\frac{5}{5}=\lim_{x\to 0}\frac{\sin 5x}{5x}\cdot 5=5\lim_{x\to 0}\frac{\sin 5x}{5x}$$

Let $y=5x$. As x approaches zero, so does y. So the limit becomes

$$5\lim_{x\to 0}\frac{\sin 5x}{5x}=5\lim_{y\to 0}\frac{\sin y}{y}=5\cdot 1=5$$

b) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x}$

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}=\lim_{x\to 0}\frac{1-\cos^2 x}{x\left(1+\cos x\right)}=\lim_{x\to 0}\frac{\sin^2 x}{x\left(1+\cos x\right)}=\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}\right)\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{1+\cos x}=1\cdot 0=0\end{aligned}$$

c) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2\left(1+\cos x\right)}=\lim_{x\to 0}\frac{\sin^2 x}{x^2\left(1+\cos x\right)}=\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}\right)\\ &= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\lim_{x\to 0}\frac{1}{1+\cos x}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}\frac{1}{1+\cos x}=1^2\cdot\frac{1}{2}=\frac{1}{2}\end{aligned}$$

d) $\displaystyle\lim_{x\to 0}\frac{1-\cos 3x}{x^2}$

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos 3x}{x^2} &= \lim_{x\to 0}\frac{1-\cos 3x}{x^2}\cdot\frac{1+\cos 3x}{1+\cos 3x}=\lim_{x\to 0}\frac{1-\cos^2 3x}{x^2\left(1+\cos 3x\right)}=\lim_{x\to 0}\frac{\sin^2 3x}{x^2\left(1+\cos 3x\right)}\\ &= \lim_{x\to 0}\left(\frac{\sin^2 3x}{x^2}\cdot\frac{1}{1+\cos 3x}\right)=\lim_{x\to 0}\frac{\sin^2 3x}{x^2}\cdot\lim_{x\to 0}\frac{1}{1+\cos 3x}=\\ &= \lim_{x\to 0}\left(\frac{\sin^2 3x}{x^2}\cdot\frac{9}{9}\right)\cdot\lim_{x\to 0}\frac{1}{1+\cos 3x}=9\cdot\lim_{x\to 0}\frac{\sin^2 3x}{9x^2}\cdot\lim_{x\to 0}\frac{1}{1+\cos 3x}\\ &= \lim_{x\to 0}\frac{\sin^2 3x}{\left(3x\right)^2}\cdot\lim_{x\to 0}\frac{1}{1+\cos 3x}=9\cdot 1\cdot\frac{1}{2}=\frac{9}{2}\end{aligned}$$

e) $\displaystyle\lim_{x\to 0}\frac{\tan x}{x}$

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x}=\lim_{x\to 0}\frac{\sin x}{x\cos x}=\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}=1\cdot 1=1$$

f) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\sin 6x}$

Solution: We will bring this limit to a form where $\dfrac{\sin x}{x}$ appears.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 5x}{\sin 6x} &= \lim_{x\to 0}\left(\frac{\sin 5x}{\sin 6x}\cdot\frac{x}{x}\right)=\lim_{x\to 0}\left(\frac{\sin 5x}{x}\cdot\frac{x}{\sin 6x}\right)=\lim_{x\to 0}\frac{\sin 5x}{x}\cdot\lim_{x\to 0}\frac{x}{\sin 6x}\\
&= \lim_{x\to 0}\left(\frac{\sin 5x}{x}\cdot\frac{5}{5}\right)\cdot\lim_{x\to 0}\left(\frac{x}{\sin 6x}\cdot\frac{6}{6}\right)=\lim_{x\to 0}\left(\frac{\sin 5x}{5x}\cdot 5\right)\cdot\lim_{x\to 0}\left(\frac{6x}{\sin 6x}\cdot\frac{1}{6}\right)\\
&= 5\lim_{x\to 0}\frac{\sin 5x}{5x}\cdot\frac{1}{6}\lim_{x\to 0}\frac{6x}{\sin 6x}=5\cdot 1\cdot\frac{1}{6}\cdot 1=\frac{5}{6}
\end{aligned}$$

g) $\lim\limits_{\theta\to\pi/4}\dfrac{\tan\theta-1}{\theta-\frac{\pi}{4}}$

This is clearly a $\dfrac{0}{0}$ type of an indeterminate. To simplify the denominator, we will introduce a new variable. Let $x=\theta-\dfrac{\pi}{4}$. As θ approaches $\dfrac{\pi}{4}$, x will approach zero. Also, solving $x=\theta-\dfrac{\pi}{4}$ for θ we get $\theta=x+\dfrac{\pi}{4}$. So our limit becomes

$$\lim_{\theta\to\pi/4}\frac{\tan\theta-1}{\theta-\frac{\pi}{4}}=\lim_{x\to 0}\frac{\tan\left(x+\frac{\pi}{4}\right)-1}{x}$$

Now the denominator is simple, but the numerator became more complex. We will expand $\tan\left(x+\dfrac{\pi}{4}\right)$ using the sum formula for tangent.

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan\left(x+\frac{\pi}{4}\right)-1}{x} &= \lim_{x\to 0}\frac{\dfrac{\tan x+\tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}}-1}{x}=\lim_{x\to 0}\frac{\dfrac{\tan x+1}{1-\tan x\cdot 1}-1}{x}=\lim_{x\to 0}\frac{\dfrac{\tan x+1}{1-\tan x}-\dfrac{1-\tan x}{1-\tan x}}{x}\\
&= \lim_{x\to 0}\frac{1}{x}\cdot\frac{\tan x+1-(1-\tan x)}{1-\tan x}=\lim_{x\to 0}\frac{1}{x}\cdot\frac{\tan x+1-1+\tan x}{1-\tan x}=\lim_{x\to 0}\frac{1}{x}\cdot\frac{2\tan x}{1-\tan x}\\
&= \lim_{x\to 0}\frac{\tan x}{x}\cdot\frac{2}{1-\tan x}=\lim_{x\to 0}\frac{\tan x}{x}\cdot\lim_{x\to 0}\frac{2}{1-\tan x}=1\cdot 2=2
\end{aligned}$$

h) $\lim\limits_{x\to 0}\dfrac{\sqrt{1-\cos x}}{x}$

This is also a $\dfrac{0}{0}$ type of an indeterminate.

Solution 1: We will start by multiplying both numerator and denominator by $\sqrt{1+\cos x}$.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{1-\cos x}}{x} &= \lim_{x\to 0}\frac{\sqrt{1-\cos x}}{x}\cdot\frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}=\lim_{x\to 0}\frac{\sqrt{1-\cos x}\sqrt{1+\cos x}}{x\sqrt{1+\cos x}}=\lim_{x\to 0}\frac{\sqrt{(1-\cos x)(1+\cos x)}}{x\sqrt{1+\cos x}}\\
&= \lim_{x\to 0}\frac{\sqrt{1-\cos^2 x}}{x\sqrt{1+\cos x}}=\lim_{x\to 0}\frac{\sqrt{\sin^2 x}}{x\sqrt{1+\cos x}}=\lim_{x\to 0}\frac{|\sin x|}{x\sqrt{1+\cos x}}
\end{aligned}$$

If the expression was simply $\dfrac{\sin x}{x\sqrt{1+\cos x}}$, then we would be in a good shape, since $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$. But we can not ignore that we have the absolute value of $\sin x$. We get rid of the absolute value sign by considering the sign of $\sin x$.

Case 1. If $x > 0$, then also $\sin x > 0$ and so $|\sin x| = \sin x$

$$\lim_{x\to 0^+} \frac{\sqrt{1-\cos x}}{x} = \lim_{x\to 0^+} \frac{|\sin x|}{x\sqrt{1+\cos x}} = \lim_{x\to 0^+} \frac{\sin x}{x\sqrt{1+\cos x}} = \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \frac{1}{\sqrt{1+\cos x}} = 1 \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Case 2. If $x < 0$, then also $\sin x < 0$ and so $|\sin x| = -\sin x$

$$\lim_{x\to 0^-} \frac{\sqrt{1-\cos x}}{x} = \lim_{x\to 0^-} \frac{|\sin x|}{x\sqrt{1+\cos x}} = \lim_{x\to 0^-} \frac{-\sin x}{x\sqrt{1+\cos x}} = -1 \cdot \lim_{x\to 0^-} \frac{\sin x}{x} \cdot \lim_{x\to 0^-} \frac{1}{\sqrt{1+\cos x}} = -1 \cdot \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

Since the left-hand side limit and the right-hand side limit are different, the two-sided limit is undefined.

Solution 2: Recall that $\sin^2 \dfrac{x}{2} = \dfrac{1-\cos x}{2}$ and so $1 - \cos x = 2\sin^2 \dfrac{x}{2}$.

$$\lim_{x\to 0} \frac{\sqrt{1-\cos x}}{x} = \lim_{x\to 0} \frac{\sqrt{2\sin^2 \frac{x}{2}}}{x} = \lim_{x\to 0} \frac{\left|\sqrt{2}\sin \frac{x}{2}\right|}{x}$$

We will need to be a little bit careful because of the absolute value. If x is positive (recall it is also very close to zero) then so is $\sin \dfrac{x}{2}$. If x is negative, so is $\sin \dfrac{x}{2}$. We will separately evaluate the left-side and right-side limits. Let us introduce the new variable $y = \dfrac{x}{2}$:

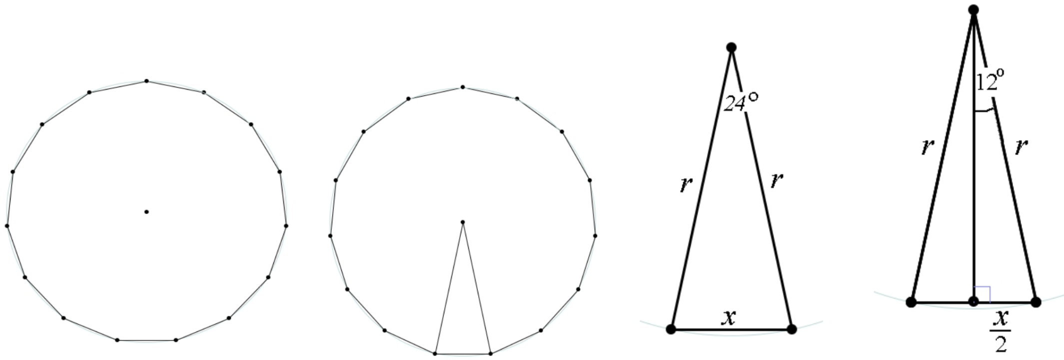
$$\begin{aligned}
\lim_{x\to 0^+} \frac{\sqrt{2}\left|\sin \frac{x}{2}\right|}{x} &= \lim_{x\to 0^+} \frac{\sqrt{2}\sin \frac{x}{2}}{x \cdot 1} = \sqrt{2} \lim_{x\to 0^+} \frac{\sin \frac{x}{2}}{x \cdot 1} = \sqrt{2} \lim_{x\to 0^+} \frac{\sin \frac{x}{2}}{x \cdot \frac{2}{2}} = \sqrt{2} \lim_{x\to 0^+} \frac{\sin \frac{x}{2}}{\frac{x}{2} \cdot 2} = \sqrt{2} \lim_{x\to 0^+} \frac{1}{2} \cdot \frac{\sin \frac{x}{2}}{\frac{x}{2}} \\
&= \frac{\sqrt{2}}{2} \lim_{y\to 0^+} \frac{\sin y}{y} = \frac{\sqrt{2}}{2} \cdot 1 = \frac{\sqrt{2}}{2}
\end{aligned}$$

The other side goes similarly:

$$\begin{aligned}
\lim_{x\to 0^-} \frac{\sqrt{2}\left|\sin \frac{x}{2}\right|}{x} &= \lim_{x\to 0^+} \frac{\sqrt{2}\left(-\sin \frac{x}{2}\right)}{x \cdot 1} = -\sqrt{2} \lim_{x\to 0^+} \frac{\sin \frac{x}{2}}{x \cdot 1} = -\sqrt{2} \lim_{x\to 0^+} \frac{\sin \frac{x}{2}}{x \cdot \frac{2}{2}} = -\sqrt{2} \lim_{x\to 0^+} \frac{\sin \frac{x}{2}}{\frac{x}{2} \cdot 2} \\
&= -\sqrt{2} \lim_{x\to 0^+} \frac{1}{2} \cdot \frac{\sin \frac{x}{2}}{\frac{x}{2}} = -\frac{\sqrt{2}}{2} \lim_{y\to 0^+} \frac{\sin y}{y} = -\frac{\sqrt{2}}{2} \cdot 1 = -\frac{\sqrt{2}}{2}
\end{aligned}$$

Since the right-hand side limit and the left-hand side limit are different, the two-sided limit is undefined.

2. a) Find the perimeter of a $15-$sided reguar polygon written into a circle with radius $10\mathrm{m}$.



The angle at the center of the circle is $\dfrac{360^\circ}{15} = 24^\circ$ If we draw the altitude belonging to side x, we create a right triangle with an angle of 12°. From this right triangle, $\sin 12^\circ = \dfrac{\frac{x}{2}}{r} = \dfrac{x}{2r}$. So $x = 2r\sin 12^\circ$. We convert the angle to radians and substitute $10\mathrm{m}$ for r. The perimeter is the sum of all 15 sides:

$$P = 15x = 15\left(2r\sin 12^\circ\right) = 30\,(10\mathrm{m})\sin\left(\frac{\pi}{15}\right) = 300\sin\left(\frac{\pi}{15}\right)\mathrm{m} \approx 62.373508\mathrm{m}$$

b) Find the perimeter of an $n-$sided regular polygon written into a circle with radius R. Use radians to measure angles.

Solution: We will perform the same steps as in the previous problem, only in the abstract. $x = 2R\sin\left(\dfrac{2\pi}{2n}\right) = 2R\sin\left(\dfrac{\pi}{n}\right)$. And so the perimeter of the polygon is

$$P = nx = n2R\sin\left(\frac{\pi}{n}\right) = 2nR\sin\left(\frac{\pi}{n}\right)$$

c) Find the limit of the perimeter of an $n-$sided regular polygon written into a circle with radius R as n approaches infinity. Use radians to measure angles.

$$\lim_{n\to\infty}\left(2nR\sin\frac{\pi}{n}\right) = 2R\lim_{n\to\infty}\left(\frac{\sin\frac{\pi}{n}}{\frac{1}{n}}\right) = 2R\lim_{n\to\infty}\left(\frac{\pi}{\pi}\cdot\frac{\sin\frac{\pi}{n}}{\frac{1}{n}}\right) = 2\pi R\lim_{n\to\infty}\left(\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right)$$

Define $x = \dfrac{\pi}{n}$. As $n\to\infty$, clearly $x\to 0$. Thus

$$2\pi R\lim_{n\to\infty}\left(\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right) = 2\pi R\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 2\pi R\cdot 1 = 2\pi R$$

3. a) Find the area of a $15-$sided regular polygon written into a circle with radius $10\mathrm{m}$.

Solution: One can compute the altitude of the right triangle using right triangle trigonometry, and compute the area that way. However, we will use a moe efficient technique. Recall that the area of a triangle can be computed as $A = \dfrac{1}{2}ab\sin\gamma$ where γ iis the angle between sides a and b. Then we can immediately compute the area of the isosceles triangle: $\dfrac{1}{2}R^2\sin 24°$. So the area of the polygon is

$$A = 15\left(\frac{1}{2}R^2\sin 24°\right) = \frac{15}{2}(10\text{m})^2\sin\left(\frac{2\pi}{15}\right) = 750\sin\left(\frac{2\pi}{15}\right)\text{m}^2 \approx 305.0525\text{m}^2$$

b) Solution: We will perform the same steps as in the previous problem, only in the abstract.

$$A = n\left(\frac{1}{2}R^2\sin\left(\frac{2\pi}{n}\right)\right) = \frac{1}{2}nR^2\sin\left(\frac{2\pi}{n}\right)$$

c) Find the limit of the area of an $n-$sided regular polygon written into a circle with radius R as n approaches infinity. Use radians to measure angles.

$$\lim_{n\to\infty}\left(\frac{1}{2}nR^2\sin\frac{2\pi}{n}\right) = \lim_{n\to\infty}\left(R^2\frac{\sin\dfrac{2\pi}{n}}{\dfrac{2}{n}}\right) = \lim_{n\to\infty}\left(\pi R^2\frac{\sin\dfrac{2\pi}{n}}{\dfrac{2\pi}{n}}\right) = \pi R^2\lim_{n\to\infty}\left(\frac{\sin\dfrac{2\pi}{n}}{\dfrac{2\pi}{n}}\right)$$

Define $x = \dfrac{2\pi}{n}$. As $n\to\infty$, clearly $x\to 0$. Thus

$$\pi R^2\lim_{n\to\infty}\left(\frac{\sin\dfrac{2\pi}{n}}{\dfrac{2\pi}{n}}\right) = \pi R^2\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \pi R^2\cdot 1 = \pi R^2$$

