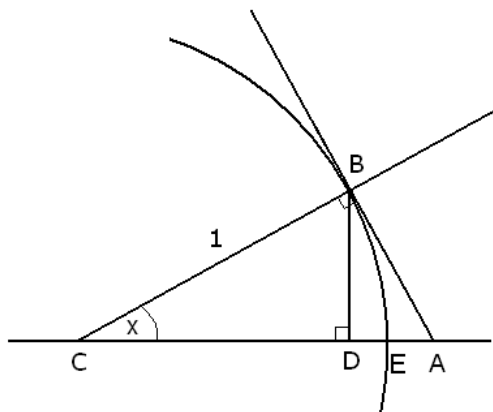


Theorem 1: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
--

Proof: This theorem and the next one are necessary for differentiating $\sin x$ and $\cos x$. Recall a theorem: Let r be the radius of a circle. If α is measured in radians, then the area of a sector with a central angle of α is $A_{\text{sector}} = \frac{\alpha r^2}{2}$. (Notation: \overline{AB} will denote the length of line segment AB .)

Let x be a very small positive angle, measured in radians, drawn into a unit circle as shown on the picture below. Let B be the point where the unit circle intersects the ray determined by x . We then draw a tangent line to the circle at point B . Let A be the point where the tangent line intersects the x -axis. We also draw a vertical line through B . Let D be the point where this vertical line intersects the x -axis. Finally, let us denote by E the point with coordinates $(0, 1)$.



The proof will be based on the following fact: because they include each other, the following three areas can be easily compared:

$$\text{Area of triangle } CDB \leq \text{Area of sector } CEB \leq \text{Area of triangle } ABC$$

Area of triangle CDB : the horizontal side, $\overline{CD} = \cos x$ and the vertical side, $\overline{DB} = \sin x$. Since this is a right triangle, the area is: $A_{CDB} = \frac{1}{2} \sin x \cos x$

Area of sector CEB : $A_{\text{sector}} = \frac{1^2 x}{2} = \frac{x}{2}$

Area of triangle ABC : there is a right angle at point B because the tangent line drawn to a circle is perpendicular to the radius drawn to the point of tangency. So the area is $A_{ABC} = \frac{1}{2} \overline{AB} \cdot \overline{BC}$. Clearly $\overline{BC} = 1$. To compute \overline{AB} , in triangle ABC , $\tan x = \frac{\overline{AB}}{1}$ and so $\overline{AB} = \tan x$.

Area of triangle ABC : $\frac{1}{2} (1) (\tan x) = \frac{\tan x}{2}$ or $\frac{\sin x}{2 \cos x}$. So now

$$\text{Area of triangle } CDB \leq \text{Area of sector } CEB \leq \text{Area of triangle } ABC$$

translates to

$$\frac{1}{2} \sin x \cos x \leq \frac{x}{2} \leq \frac{\sin x}{2 \cos x}$$

Let us divide all three sides by $\frac{\sin x}{2}$. Because x is small and positive, $\frac{\sin x}{2}$ is positive and so we do not need to reverse the inequality signs.

$$\cos x \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

Suppose now that x approaches zero. Then both $\cos x$ and $\frac{1}{\cos x}$ approach 1. By the sandwich principle, $\frac{x}{\sin x}$, the quantity locked in between those two must also approach 1.

$$\begin{array}{ccc} \cos x & \leq & \frac{x}{\sin x} \leq \frac{1}{\cos x} \\ \downarrow & & \downarrow \\ 1 & & 1 \end{array}$$

If $\frac{x}{\sin x}$ approaches 1, so does its reciprocal, $\frac{\sin x}{x}$.

So far, we have proven the statement for positive values of x , that is, $\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$. A similar argument works for negative values of x .

Theorem 2: $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$

Proof:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} &= \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} \cdot 1 = \lim_{x \rightarrow 0} \left(\frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} \right) = \lim_{x \rightarrow 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x \rightarrow 0} \frac{-(1 - \cos^2 x)}{x(\cos x + 1)} \\ &= \lim_{x \rightarrow 0} \frac{-\sin^2 x}{x(\cos x + 1)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{-\sin x}{\cos x + 1} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{-\sin x}{\cos x + 1} = 1 \cdot 0 = 0 \end{aligned}$$

Sample Problems

1. Recall that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. Use this fact to compute each of the following limits.

a) $\lim_{x \rightarrow 0} \frac{\sin 5x}{x}$

c) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$

f) $\lim_{\theta \rightarrow \pi/4} \frac{\tan \theta - 1}{\theta - \frac{\pi}{4}}$

b) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$

d) $\lim_{x \rightarrow 0} \frac{\tan x}{x}$

e) $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 6x}$

g) $\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos x}}{x}$

2. a) Find the perimeter of a 15-sided regular polygon written into a circle with radius 10 m.
 b) Find the perimeter of an n -sided regular polygon written into a circle with radius R . Use radians to measure angles.
 c) Find the limit of the perimeter of an n -sided regular polygon written into a circle with radius R as n approaches infinity. Use radians to measure angles.
3. a) Find the area of a 15-sided regular polygon written into a circle with radius 10 m.
 b) Find the area of an n -sided regular polygon written into a circle with radius R . Use radians to measure angles.
 c) Find the limit of the area of an n -sided regular polygon written into a circle with radius R as n approaches infinity. Use radians to measure angles.

Practice Problems

Compute each of the following limits.

1. $\lim_{x \rightarrow 0} \frac{\sin 5x}{x}$
2. $\lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{x}$
3. $\lim_{\theta \rightarrow 0} \frac{\cos 3\theta \sin 3\theta}{\theta}$
4. $\lim_{x \rightarrow 0} \frac{\sin 2x + \sin 4x}{x}$
5. $\lim_{\theta \rightarrow 0} \frac{\tan 4\theta}{5\theta}$
6. $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 3x}$
7. $\lim_{\theta \rightarrow 0} \frac{\theta}{\sin 3\theta}$
8. $\lim_{x \rightarrow 0} \frac{\sin^2 x}{3x^2}$
9. $\lim_{\theta \rightarrow 0} \frac{\sin(\theta^2)}{\theta \tan \theta}$
10. $\lim_{x \rightarrow 0} \frac{\tan 6x}{3x}$
11. $\lim_{x \rightarrow 0} \frac{\sin x \tan x}{x^2}$
12. $\lim_{x \rightarrow 0} \frac{\sin 2x \tan 3x}{x^2}$
13. $\lim_{\theta \rightarrow 0} \frac{\theta \sin 2\theta}{2 - 2 \cos^2 \theta}$
14. $\lim_{x \rightarrow 0} \frac{x}{\sin 2x}$
15. $\lim_{x \rightarrow 0} \frac{\tan x}{\tan 4x}$
16. $\lim_{x \rightarrow 0} \frac{6x}{\sin 4x + \sin 3x}$
17. $\lim_{\theta \rightarrow 0} \frac{2\theta^2}{1 - \cos \theta}$
18. $\lim_{x \rightarrow \pi/2} \frac{\sin x - 1}{x - \frac{\pi}{2}}$
19. $\lim_{\theta \rightarrow 0} \frac{\theta \sin \theta}{1 - \cos \theta}$

Sample Problems - Answers

1. a) 5 b) 0 c) $\frac{1}{2}$ d) 1 e) $\frac{5}{6}$ f) 2 g) undefined
2. a) $300 \sin\left(\frac{\pi}{15}\right) \text{ m} \approx 62.3735 \text{ m}$ b) $2nR \sin \frac{\pi}{n}$ c) $2\pi R$
3. a) $750 \sin\left(\frac{2\pi}{15}\right) \text{ m}^2 \approx 305.0525 \text{ m}^2$ b) $\frac{1}{2}nR^2 \sin \frac{2\pi}{n}$ c) πR^2

Answers - Practice Problems

- 1.) 5 2.) 2 3.) 3 4.) 6 5.) $\frac{4}{5}$ 6.) $\frac{4}{3}$ 7.) $\frac{1}{3}$ 8.) $\frac{1}{3}$ 9.) 1 10.) 2
- 11.) 1 12.) 6 13.) 1 14.) $\frac{1}{2}$ 15.) $\frac{1}{4}$ 16.) $\frac{6}{7}$ 17.) 4 18.) 0 19.) 2

Sample Problems - Solutions

1. Recall that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. Use this fact to compute each of the following limits.

a) $\lim_{x \rightarrow 0} \frac{\sin 5x}{x}$

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{x} = \lim_{x \rightarrow 0} \frac{\sin 5x}{x} \cdot 1 = \lim_{x \rightarrow 0} \frac{\sin 5x}{x} \cdot \frac{5}{5} = \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \cdot 5 = 5 \lim_{x \rightarrow 0} \frac{\sin 5x}{5x}$$

Let $y = 5x$. As x approaches zero, so does y . So the limit becomes

$$5 \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} = 5 \lim_{y \rightarrow 0} \frac{\sin y}{y} = 5 \cdot 1 = 5$$

b) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} = \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} \right) \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} = 1 \cdot 0 = 0 \end{aligned}$$

c) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x} = \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2(1 + \cos x)} = \lim_{x \rightarrow 0} \left(\frac{\sin^2 x}{x^2} \cdot \frac{1}{1 + \cos x} \right) \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \cdot \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} = \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^2 \cdot \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} = 1^2 \cdot \frac{1}{2} = \frac{1}{2} \end{aligned}$$

d) $\lim_{x \rightarrow 0} \frac{\tan x}{x}$

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x \cos x} = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1 \cdot 1 = 1$$

e) $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 6x}$

Solution: We will bring this limit to a form where $\frac{\sin x}{x}$ appears.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 6x} &= \lim_{x \rightarrow 0} \left(\frac{\sin 5x}{\sin 6x} \cdot \frac{x}{x} \right) = \lim_{x \rightarrow 0} \left(\frac{\sin 5x}{x} \cdot \frac{x}{\sin 6x} \right) = \lim_{x \rightarrow 0} \frac{\sin 5x}{x} \cdot \lim_{x \rightarrow 0} \frac{x}{\sin 6x} \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin 5x}{x} \cdot \frac{5}{5} \right) \cdot \lim_{x \rightarrow 0} \left(\frac{x}{\sin 6x} \cdot \frac{6}{6} \right) = \lim_{x \rightarrow 0} \left(\frac{\sin 5x}{5x} \cdot 5 \right) \cdot \lim_{x \rightarrow 0} \left(\frac{6x}{\sin 6x} \cdot \frac{1}{6} \right) \\ &= 5 \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \cdot \frac{1}{6} \lim_{x \rightarrow 0} \frac{6x}{\sin 6x} = 5 \cdot 1 \cdot \frac{1}{6} \cdot 1 = \frac{5}{6} \end{aligned}$$

$$f) \lim_{\theta \rightarrow \pi/4} \frac{\tan \theta - 1}{\theta - \frac{\pi}{4}}$$

This is clearly a $\frac{0}{0}$ type of an indeterminate. To simplify the denominator, we will introduce a new variable.

Let $x = \theta - \frac{\pi}{4}$. As θ approaches $\frac{\pi}{4}$, x will approach zero. Also, solving $x = \theta - \frac{\pi}{4}$ for θ we get $\theta = x + \frac{\pi}{4}$. So our limit becomes

$$\lim_{\theta \rightarrow \pi/4} \frac{\tan \theta - 1}{\theta - \frac{\pi}{4}} = \lim_{x \rightarrow 0} \frac{\tan \left(x + \frac{\pi}{4}\right) - 1}{x}$$

Now the denominator is simple, but the numerator became more complex. We will expand $\tan \left(x + \frac{\pi}{4}\right)$ using the sum formula for tangent.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan \left(x + \frac{\pi}{4}\right) - 1}{x} &= \lim_{x \rightarrow 0} \frac{\frac{\tan x + \tan \frac{\pi}{4}}{1 - \tan x \tan \frac{\pi}{4}} - 1}{x} = \lim_{x \rightarrow 0} \frac{\frac{\tan x + 1}{1 - \tan x \cdot 1} - 1}{x} = \lim_{x \rightarrow 0} \frac{\frac{\tan x + 1}{1 - \tan x} - \frac{1 - \tan x}{1 - \tan x}}{x} \\ &= \lim_{x \rightarrow 0} \frac{1}{x} \cdot \frac{\tan x + 1 - (1 - \tan x)}{1 - \tan x} = \lim_{x \rightarrow 0} \frac{1}{x} \cdot \frac{\tan x + 1 - 1 + \tan x}{1 - \tan x} = \lim_{x \rightarrow 0} \frac{1}{x} \cdot \frac{2 \tan x}{1 - \tan x} \\ &= \lim_{x \rightarrow 0} \frac{\tan x}{x} \cdot \frac{2}{1 - \tan x} = \lim_{x \rightarrow 0} \frac{\tan x}{x} \cdot \lim_{x \rightarrow 0} \frac{2}{1 - \tan x} = 1 \cdot 2 = 2 \end{aligned}$$

$$g) \lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos x}}{x}$$

This is also a $\frac{0}{0}$ type of an indeterminate.

Solution 1: We will start by multiplying both numerator and denominator by $\sqrt{1 + \cos x}$.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos x}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos x}}{x} \cdot \frac{\sqrt{1 + \cos x}}{\sqrt{1 + \cos x}} = \lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos x} \sqrt{1 + \cos x}}{x \sqrt{1 + \cos x}} = \lim_{x \rightarrow 0} \frac{\sqrt{(1 - \cos x)(1 + \cos x)}}{x \sqrt{1 + \cos x}} \\ &= \lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos^2 x}}{x \sqrt{1 + \cos x}} = \lim_{x \rightarrow 0} \frac{\sqrt{\sin^2 x}}{x \sqrt{1 + \cos x}} = \lim_{x \rightarrow 0} \frac{|\sin x|}{x \sqrt{1 + \cos x}} \end{aligned}$$

If the expression was simply $\frac{\sin x}{x \sqrt{1 + \cos x}}$, then we would be in a good shape, since $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. But we can not ignore that we have the absolute value of $\sin x$. We get rid of the absolute value sign by considering the sign of $\sin x$.

Case 1. If $x > 0$, then also $\sin x > 0$ and so $|\sin x| = \sin x$

$$\lim_{x \rightarrow 0^+} \frac{\sqrt{1 - \cos x}}{x} = \lim_{x \rightarrow 0^+} \frac{|\sin x|}{x \sqrt{1 + \cos x}} = \lim_{x \rightarrow 0^+} \frac{\sin x}{x \sqrt{1 + \cos x}} = \lim_{x \rightarrow 0^+} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{1 + \cos x}} = 1 \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Case 2. If $x < 0$, then also $\sin x < 0$ and so $|\sin x| = -\sin x$

$$\lim_{x \rightarrow 0^-} \frac{\sqrt{1 - \cos x}}{x} = \lim_{x \rightarrow 0^-} \frac{|\sin x|}{x \sqrt{1 + \cos x}} = \lim_{x \rightarrow 0^-} \frac{-\sin x}{x \sqrt{1 + \cos x}} = -1 \cdot \lim_{x \rightarrow 0^-} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0^-} \frac{1}{\sqrt{1 + \cos x}} = -1 \cdot \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

Since the left-hand side limit and the right-hand side limit are different, the two-sided limit is undefined.

Solution 2: Recall that $\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$ and so $1 - \cos x = 2 \sin^2 \frac{x}{2}$.

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos x}}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{2 \sin^2 \frac{x}{2}}}{x} = \lim_{x \rightarrow 0} \frac{|\sqrt{2} \sin \frac{x}{2}|}{x}$$

We will need to be a little bit careful because of the absolute value. If x is positive (recall it is also very close to zero) then so is $\sin \frac{x}{2}$. If x is negative, so is $\sin \frac{x}{2}$. We will separately evaluate the left-side and right-side limits. Let us introduce the new variable $y = \frac{x}{2}$:

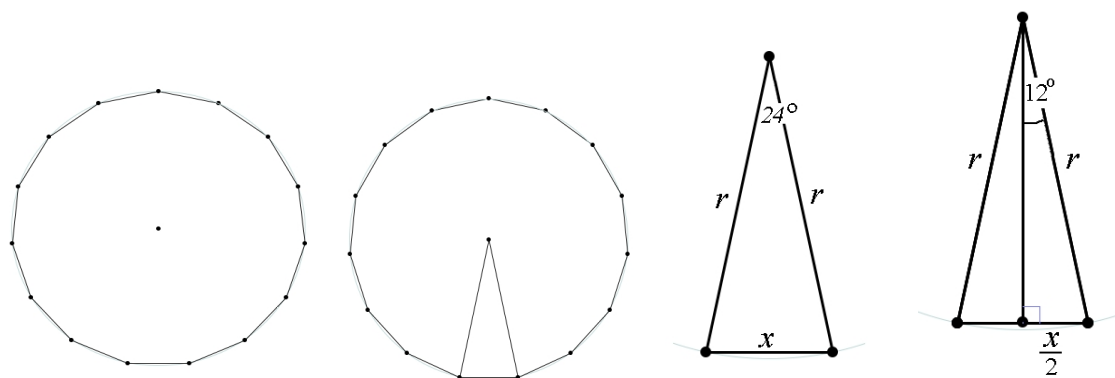
$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\sqrt{2} \left| \sin \frac{x}{2} \right|}{x} &= \lim_{x \rightarrow 0^+} \frac{\sqrt{2} \sin \frac{x}{2}}{x \cdot 1} = \sqrt{2} \lim_{x \rightarrow 0^+} \frac{\sin \frac{x}{2}}{x \cdot 1} = \sqrt{2} \lim_{x \rightarrow 0^+} \frac{\sin \frac{x}{2}}{x \cdot \frac{2}{2}} = \sqrt{2} \lim_{x \rightarrow 0^+} \frac{\sin \frac{x}{2}}{\frac{x}{2} \cdot 2} = \sqrt{2} \lim_{x \rightarrow 0^+} \frac{1}{2} \cdot \frac{\sin \frac{x}{2}}{\frac{x}{2}} \\ &= \frac{\sqrt{2}}{2} \lim_{y \rightarrow 0^+} \frac{\sin y}{y} = \frac{\sqrt{2}}{2} \cdot 1 = \frac{\sqrt{2}}{2} \end{aligned}$$

The other side goes similarly:

$$\begin{aligned} \lim_{x \rightarrow 0^-} \frac{\sqrt{2} \left| \sin \frac{x}{2} \right|}{x} &= \lim_{x \rightarrow 0^+} \frac{\sqrt{2} \left(-\sin \frac{x}{2} \right)}{x \cdot 1} = -\sqrt{2} \lim_{x \rightarrow 0^+} \frac{\sin \frac{x}{2}}{x \cdot 1} = -\sqrt{2} \lim_{x \rightarrow 0^+} \frac{\sin \frac{x}{2}}{x \cdot \frac{2}{2}} = -\sqrt{2} \lim_{x \rightarrow 0^+} \frac{\sin \frac{x}{2}}{\frac{x}{2} \cdot 2} = -\sqrt{2} \lim_{x \rightarrow 0^+} \frac{1}{2} \cdot \frac{\sin \frac{x}{2}}{\frac{x}{2}} \\ &= -\sqrt{2} \lim_{x \rightarrow 0^+} \frac{1}{2} \cdot \frac{\sin \frac{x}{2}}{\frac{x}{2}} = -\frac{\sqrt{2}}{2} \lim_{y \rightarrow 0^+} \frac{\sin y}{y} = -\frac{\sqrt{2}}{2} \cdot 1 = -\frac{\sqrt{2}}{2} \end{aligned}$$

Since the right-hand side limit and the left-hand side limit are different, the two-sided limit is undefined.

2. a) Find the perimeter of a 15-sided regular polygon written into a circle with radius 10 m.



The angle at the center of the circle is $\frac{360^\circ}{15} = 24^\circ$. If we draw the altitude belonging to side x , we create a right triangle with an angle of 12° . From this right triangle, $\sin 12^\circ = \frac{\frac{x}{2}}{r} = \frac{x}{2r}$. So $x = 2r \sin 12^\circ$. We convert the angle to radians and substitute 10 m for r . The perimeter is the sum of all 15 sides:

$$P = 15x = 15(2r \sin 12^\circ) = 30(10 \text{ m}) \sin \left(\frac{\pi}{15} \right) = 300 \sin \left(\frac{\pi}{15} \right) \text{ m} \approx 62.373508 \text{ m}$$

b) Find the perimeter of an n -sided regular polygon written into a circle with radius R . Use radians to measure angles.

Solution: We will perform the same steps as in the previous problem, only in the abstract. $x = 2R \sin\left(\frac{2\pi}{2n}\right) = 2R \sin\left(\frac{\pi}{n}\right)$. And so the perimeter of the polygon is

$$P = nx = n2R \sin\left(\frac{\pi}{n}\right) = 2nR \sin\left(\frac{\pi}{n}\right)$$

c) Find the limit of the perimeter of an n -sided regular polygon written into a circle with radius R as n approaches infinity. Use radians to measure angles.

$$\lim_{n \rightarrow \infty} \left(2nR \sin \frac{\pi}{n}\right) = 2R \lim_{n \rightarrow \infty} \left(\frac{\sin \frac{\pi}{n}}{\frac{1}{n}}\right) = 2R \lim_{n \rightarrow \infty} \left(\frac{\pi}{\pi} \cdot \frac{\sin \frac{\pi}{n}}{\frac{1}{n}}\right) = 2\pi R \lim_{n \rightarrow \infty} \left(\frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}}\right)$$

Define $x = \frac{\pi}{n}$. As $n \rightarrow \infty$, clearly $x \rightarrow 0$. Thus

$$2\pi R \lim_{n \rightarrow \infty} \left(\frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}}\right) = 2\pi R \lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right) = 2\pi R \cdot 1 = 2\pi R$$

3. a) Find the area of a 15-sided regular polygon written into a circle with radius 10 m.

Solution: One can compute the altitude of the right triangle using right triangle trigonometry, and compute the area that way. However, we will use a more efficient technique. Recall that the area of a triangle can be computed as $A = \frac{1}{2}ab \sin \gamma$ where γ is the angle between sides a and b . Then we can immediately compute the area of the isosceles triangle: $\frac{1}{2}R^2 \sin 24^\circ$. So the area of the polygon is

$$A = 15 \left(\frac{1}{2}R^2 \sin 24^\circ\right) = \frac{15}{2} (10 \text{ m})^2 \sin\left(\frac{2\pi}{15}\right) = 750 \sin\left(\frac{2\pi}{15}\right) \text{ m}^2 \approx 305.0525 \text{ m}^2$$

b) Solution: We will perform the same steps as in the previous problem, only in the abstract.

$$A = n \left(\frac{1}{2}R^2 \sin\left(\frac{2\pi}{n}\right)\right) = \frac{1}{2}nR^2 \sin\left(\frac{2\pi}{n}\right)$$

c) Find the limit of the area of an n -sided regular polygon written into a circle with radius R as n approaches infinity. Use radians to measure angles.

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2}nR^2 \sin \frac{2\pi}{n}\right) = \lim_{n \rightarrow \infty} \left(R^2 \frac{\sin \frac{2\pi}{n}}{\frac{2}{n}}\right) = \lim_{n \rightarrow \infty} \left(\pi R^2 \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}\right) = \pi R^2 \lim_{n \rightarrow \infty} \left(\frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}\right)$$

Define $x = \frac{2\pi}{n}$. As $n \rightarrow \infty$, clearly $x \rightarrow 0$. Thus

$$\pi R^2 \lim_{n \rightarrow \infty} \left(\frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}\right) = \pi R^2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right) = \pi R^2 \cdot 1 = \pi R^2$$

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