We will apply derivatives to find maximums and minimums of functions beyond quadratics. There are two kinds of maximums/minimums: relative and absolute.

Definition: Extremum is a common name for a maximum or a minimum. A function f has an **absolute maximum** at x_M if for all x in the domain, $f(x_M) \geq f(x)$. A function f has an **absolute minimum** at x_m if for all x in the domain, $f(x_m) \leq f(x)$. A function f has a **relative (or local) maximum** at x_M if there exists an open interval $I = (a, b)$ that contains x_M such that
i) the function is defined on I and
ii) if we restrict the function to I as its domain, then $(x_M, f(x_M))$ is an absolute maximum.

A function f has a **relative (or local) minimum** at x_m if there exists an open interval $I = (a, b)$ that contains x_m such that
i) the function is defined on I and
ii) if we restrict the function to I as its domain, then $(x_m, f(x_m))$ is an absolute minimum.

Definition: A function f is **increasing** on an interval I if for all a and b in I, if $a < b$, then $f(a) \leq f(b)$. A function f is **strictly increasing** on an interval I if for all a and b in I, if $a < b$, then $f(a) < f(b)$.

Definition: A function f is **decreasing** on an interval I if for all a and b in I, if $a < b$, then $f(a) \geq f(b)$. A function f is **strictly decreasing** on an interval I if for all a and b in I, if $a < b$, then $f(a) > f(b)$.

Recall two helpful way we can think of a fuction f and its derivative, f'

1. We can always interpret f and f' as location and velocity functions of the same object.
2. The derivative measures the slope of the tangent line.

If f is increasing on I, then $f'(x)$ (if exists,) is non-negative, i.e $f'(x) \geq 0$. Using these two ideas, we can explain this connection between properties of f and f'.

1. If the location function is increasing, that means that the object is moving upward. Then its velocity is positive.
2. If f is increasing, the tangent line drawn to it also increases. Therefore, the tangent line has a positive slope.

A relatie maximum is where f changes behavior from increasing to decreasing. A relative minimum is where f changes behavior from decreasing to increasing. Therefore, if the derivative exists, it will change sign. If a function changes sign while continuous, it will take a zero value.

Theorem: (First Derivative Theorem for Local Extreme Values) If f is has a relative maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

Proof: Suppose that f is differentiable at c and f has a local maximum at c. Then $f'(c) = \lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ exists and is a two-sided limit. For a sufficiently small positive value of h, $f(c+h)$ exists and $f(c+h) \leq f(c)$ since f has a local maximum value at c. Then $f(c+h) - f(c) \leq 0$. Divide that by positive h and get that $\dfrac{f(c+h) - f(c)}{h} \leq 0$ and so

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

Now let h be a very small negative number. Then by the same argument, $f(c+h) \leq f(c)$. Divide that by a negative h and get that $\dfrac{f(c+h) - f(c)}{h} \geq 0$ and so

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

For the two-sided limit $f'(c)$ to exists, we must have

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

Since one side is less than or equal to zero and the other is greater than or equal to zero, they both must be zero. This completes our proof.

So, if the derivative exists at a relative maximum, it must be zero. Notice that this does not mean that every zero of the derivative is at a local extrema. For example, the derivative of $f(x) = x^3$ is $f'(x) = 3x^2$, and it takes a zero at $x = 0$. Yet, the zero of the derivative does not indicate a maximum or minimum, because there is no change in sign in the derivative. $3x^2$ is positive at both soides of its zero, indicating that x^3 is increasing. For a maximum, the derivative should change its sign from positive to negative.

To find relative extrema of a function f, we differentiate f, find the zeroes of f' and look around the them for change in the sign of f'.

Sample Problems

For each of the given functions f,

a) Find all values of x for which f is increasing.

b) Find all values of x for which f is decreasing.

c) Find all values of x for which f has a relative maximum.

d) Find all values of x for which f has a relative minimum.

e) Sketch the graph of f.

1. $f(x) = -2x^3 + 9x^2 + 24$.

2. $f(x) = 6x^5 - 50x^3 - 120$.

Practice Problems

For each of the given functions f,

a) Find all values of x for which f is increasing.

b) Find all values of x for which f is decreasing.

c) Find all values of x for which f has a relative maximum.

d) Find all values of x for which f has a relative minimum.

e) Sketch the graph of f.

1. $f(x) = x^3 - 3x^2 + 6$
2. $f(x) = -x^3 + 6x^2 + 36x - 60$
3. $f(x) = -2x^3 + 12x^2 + 6x - 24$
4. $f(x) = 3x^4 - 6x^2 + 8$
5. $f(x) = -6x^5 + 20x^3$
6. $f(x) = 5x^6 - 24x^5 + 30x^4 - 12$
7. $f(x) = -3x^4 + 16x^3 + 1$
8. $f(x) = -x^6 + 6x^4$

Answers

Sample Problems

1. $f(x) = -2x^3 + 9x^2 + 24$

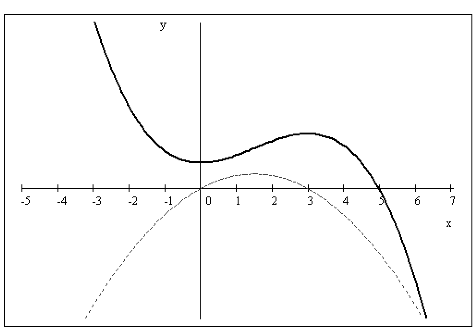
   a) increasing on $(0, 3)$

   b) decreasing on $(-\infty, 0)$ and on $(3, \infty)$

   c) relative maximum at $x = 3$

   d) relative minimum at $x = 0$

   e)



2. $f(x) = 6x^5 - 50x^3 - 120$

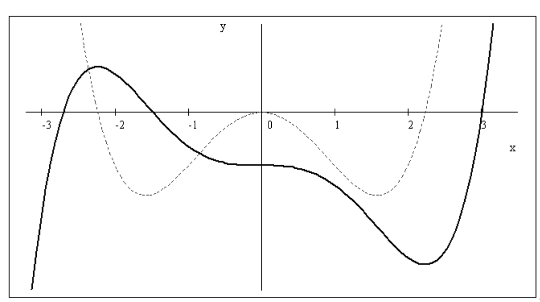
   a) increasing on $\left(-\infty, -\sqrt{5}\right)$ and on $\left(\sqrt{5}, \infty\right)$

   b) decreasing on $\left(-\sqrt{5}, \sqrt{5}\right)$

   c) relative maximum at $x = -\sqrt{5}$

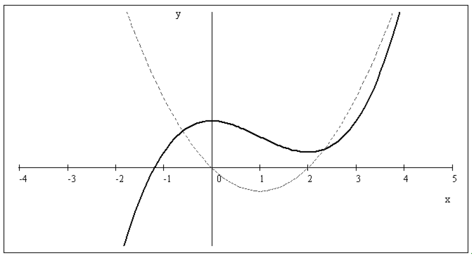
   d) relative minimum at $x = \sqrt{5}$

   e)

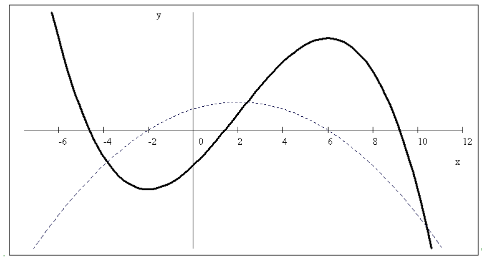


Answers for Practice Problems

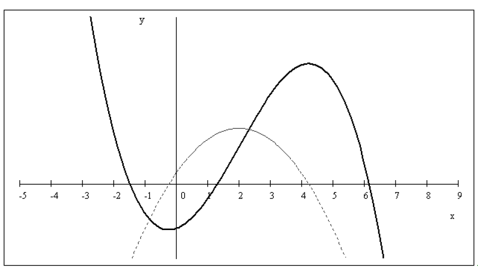
1. $f(x) = x^3 - 3x^2 + 6$
   a) increasing on $(-\infty, 0)$ and on $(2, \infty)$
   b) decreasing on $(0, 2)$
   c) relative maximum at $x = 0$
   d) relative minimum at $x = 2$
   e)



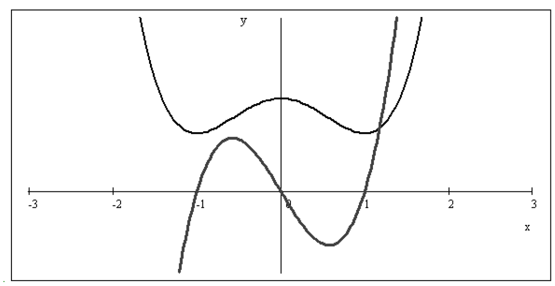
2. $f(x) = -x^3 + 6x^2 + 36x - 60$
   a) increasing on $(-2, 6)$
   b) decreasing on $(-\infty, -2)$ and on $(6, \infty)$
   c) relative maximum at $x = 6$
   d) relative minimum at $x = -2$
   e)



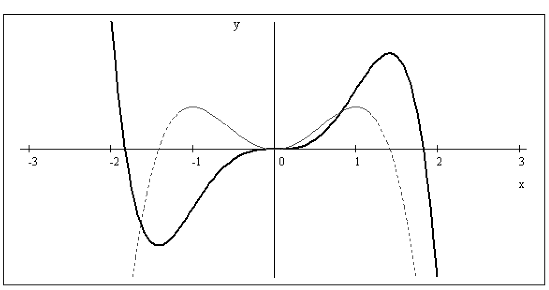
3. $f(x) = -2x^3 + 12x^2 + 6x - 24$
   a) increasing on $\left(2 - \sqrt{5}, 2 + \sqrt{5}\right)$
   b) decreasing on $\left(-\infty, 2 - \sqrt{5}\right)$ and on $\left(2 + \sqrt{5}, \infty\right)$
   c) relative maximum at $x = 2 + \sqrt{5}$
   d) relative minimum at $x = 2 - \sqrt{5}$
   e)



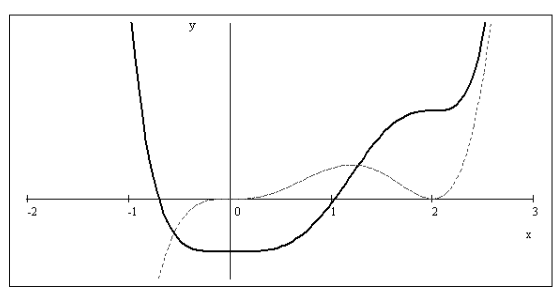
4. $f(x) = 3x^4 - 6x^2 + 8$
   a) increasing on $(-1, 0)$ and on $(1, \infty)$
   b) decreasing on $(-\infty, -1)$ and on $(0, 1)$
   c) relative maximum at $x = 0$
   d) relative minimum at $x = -1, 1$
   e)



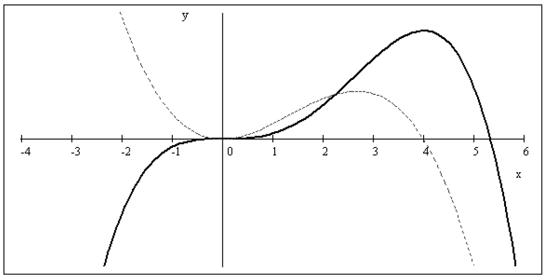
5. $f(x) = -6x^5 + 20x^3$
   a) increasing on $\left(-\sqrt{2}, 0\right)$ and on $\left(0, \sqrt{2}\right)$
   b) decreasing on $\left(-\infty, -\sqrt{2}\right)$ and on $\left(\sqrt{2}, \infty\right)$
   c) relative maximum at $x = \sqrt{2}$
   d) relative minimum at $x = -\sqrt{2}$
   e)



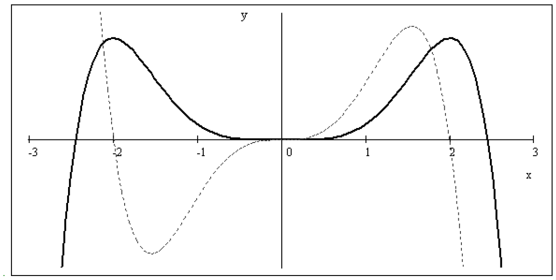
6. $f(x) = 5x^6 - 24x^5 + 30x^4 - 12$
   a) increasing on $(0, \infty)$
   b) decreasing on $(-\infty, 0)$
   c) no relative maximum
   d) relative minimum at $x = 0$
   e)



7. $f(x) = -3x^4 + 16x^3 + 1$
   a) increasing on $(-\infty, 4)$
   b) decreasing on $(4, \infty)$
   c) relative maximum at $x = 4$
   d) no relative minimum
   e)



8. $f(x) = -x^6 + 6x^4$
   a) increasing on $(-\infty, -2)$ and on $(0, 2)$
   b) decreasing on $(-2, 0)$ and on $(2, \infty)$
   c) relative maximum at $x = -2, 2$
   d) relative minimum at $x = 0$
   e)



Sample Problems - Solutions

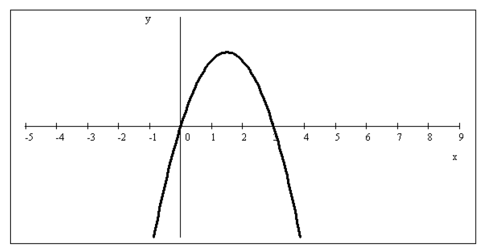
1. Consider the function $f(x) = -2x^3 + 9x^2 + 24$.

a) Find all values of x for which f is increasing.

Solution: We compute f' first and then determine when f' is positive.

$$\begin{aligned} f(x) &= -2x^3 + 9x^2 + 24 \\ f'(x) &= -6x^2 + 18x = -6x(x-3) \end{aligned}$$

f' is a quadratic function with a negative leading coefficient. Therefore, its graph is a downward opening parabola. The $x-$intercepts are at $x = 0$ and $x = 3$. When f' is positive, then f is increasing. We can see that f' is positive on $(0, 3)$ and so f is increasing there. The answer is: f is increasing on $(0, 3)$.



$f'(x) = -6x(x-3)$

b) Find all values of x for which f is decreasing.

Solution: we differentiate f, and then factor and graph f' (see in part a). When f' is negative, then f is decreasing. From the graph we can see that f' is negative on $(-\infty, 0)$ and $(3, \infty)$ and so f is decreasing there. The answer is: f is decreasing on $(-\infty, 0)$ and on $(3, \infty)$.

c) Find all values of x for which f has a relative maximum.

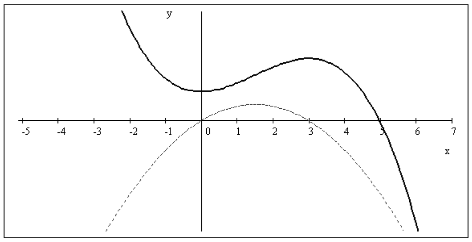
Solution: Since f is a polynomial, it is continuous everywhere. Thus f has a relative maximum at x if f changes from increasing to decreasing at x. That happens when f' changes sign from positive to negative. Based on the answers for parts a) and b), this happens at $x = 3$. Thus f has a relative maximum at $x = 3$.

d) Find all values of x for which f has a relative minimum.

Solution: Since f is a polynomial, it is continuous everywhere. Thus f has a relative minimum at x if f changes from decreasing to increasing at x. That is the same as f' changing from negative to positive. Based on the answers for parts a) and b), this happens at $x = 0$. Thus f has a relative minimum at $x = 0$.

e) Sketch the graph of f.

Solution: We evaluate f at the relative minimum and maximum.
$f(0) = 24$ and $f(3) = 51$. Note that the only $x-$intercept of f is irrational and it would take solving a cubic equation to find its exact value. If needed, we may evaluate f at additional points. In case of polynomials, the $y-$intercept is an easy one.

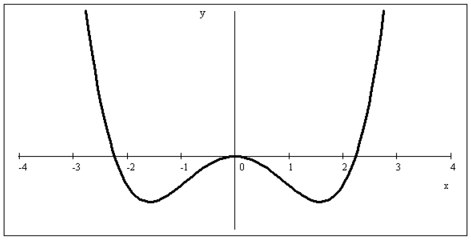


2. Consider the function $f(x) = 6x^5 - 50x^3 - 120$.

a) Find all values of x for which f is increasing.

Solution: Let us compute f' first.

$$\begin{aligned} f(x) &= 6x^5 - 50x^3 - 120 \\ f'(x) &= 30x^4 - 150x^2 = 30x^2\left(x^2 - 5\right) = 30x^2\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right) \end{aligned}$$



$f'(x) = x^2\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$

f' is a degree four polynomial function with a positive leading coefficient and $x-$intercepts at $x = -\sqrt{5}, 0, \sqrt{5}$. When f' is positive, then f is increasing. We can see that f' is positive on $\left(-\infty, -\sqrt{5}\right)$ and $\left(\sqrt{5}, \infty\right)$ so f is increasing there. The answer is: f is increasing on $\left(-\infty, -\sqrt{5}\right)$ and on $\left(\sqrt{5}, \infty\right)$.

b) Find all values of x for which f is decreasing.

Solution: we differentiate f, and then factor and graph f' (see in part a). When f' is negative, then f is decreasing. From the graph we can see that f' is negative on $\left(-\sqrt{5}, 0\right)$ and on $\left(0, \sqrt{5}\right)$ and so f is decreasing there.
In this particular case, f is decreasing on the entire interval $\left(-\sqrt{5}, \sqrt{5}\right)$. One must be careful when making this step. For example, the function $g(x) = \dfrac{1}{x}$ is decreasing on $(-\infty, 0)$ and $(0, \infty)$ but is not decreasing on $(-\infty, \infty)$.

c) Find all values of x for which f has a relative maximum.

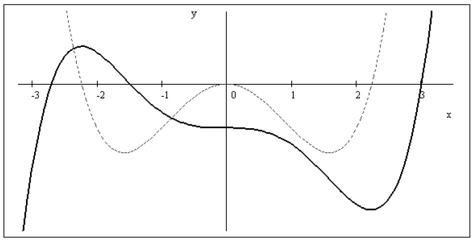
Solution: Since f is a polynomial, it is continuous everywhere. Thus f has a relative maximum at x if f changes from increasing to decreasing at x. That happens when f' changes sign from positive to negative. Based on the answers for parts a) and b), this happens at $x = -\sqrt{5}$. Thus f has a relative maximum at $x = -\sqrt{5}$.

d) Find all values of x for which f has a relative minimum.

Solution: Since f is a polynomial, it is continuous everywhere. Thus f has a relative minimum at x if f changes from decreasing to increasing at x. That happens when f' changes sign from negative to positive. Based on the answers for parts a) and b), this happens at $x = \sqrt{5}$. Thus f has a relative minimum at $x = \sqrt{5}$.
What about the zero of f' at $x = 0$? Does f have a relative maximum or a minimum at $x = 0$? The answer is: neither. The derivative f' is negative on an interval before $x = 0$ and also on an interval after $x = 0$. Consequently, f is decreasing before and after $x = 0$ and so there can be neither a relative maximum nor a relative minimum there. This situation is similar to $g(x) = x^3$ at $x = 0$. Although its derivative, $g'(x) = 3x^2$ has a zero at $x = 0$, there is no change in sign around the zero there, and so $g(x) = x^3$ does not have a relative minimum or maximum at $x = 0$.

e) Sketch the graph of f.

Solution: We evaluate f at the relative minimum and maximum.
$f(0) = -120$, $f\left(-\sqrt{5}\right) = -120 + 100\sqrt{5} \approx 103.6068$ and $f\left(\sqrt{5}\right) = -120 - 100\sqrt{5} \approx -343.6068$.



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