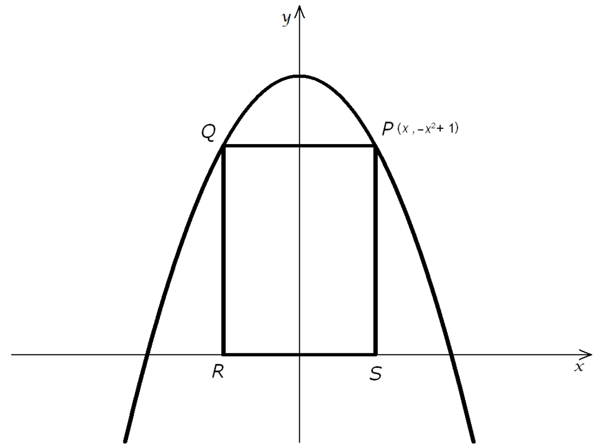
Sample Problems

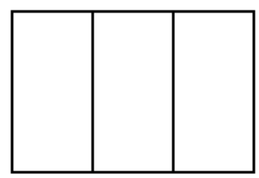
1. Consider $f(x) = -x^4 + 4x^2 + 2$ on the interval $[-3, 3]$.
   a) Find all values of x for which f has a relative maximum.
   b) Find all values of x for which f has a relative minimum.
   c) Find all absolute maximums and minimums of f.

2. Let $P(x, y)$ be a point on the graph of $y = -x^2 + 1$ with $0 \leq x \leq 1$. Let $PQRS$ be a rectangle with one side on the $x-$axis and two vertices on the graph, as shown on the picture next. Find the exact value of the greatest possible area of such a rectangle.

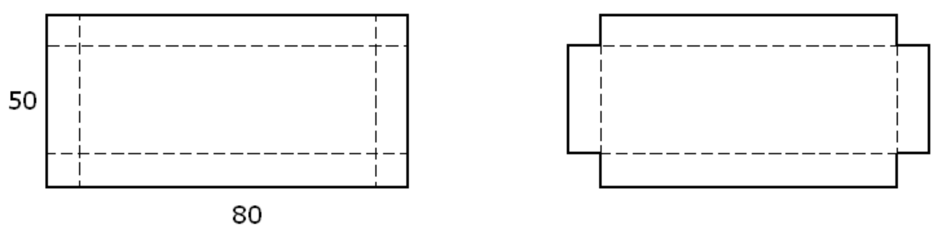


3. One positive number plus the square of another equals 48. Choose the numbers so that their product is as large as possible.

4. We have P meters of fencing and want to create three adjacent rectangular enclosings as shown on the figure. What is the maximal area we can enclose this way?



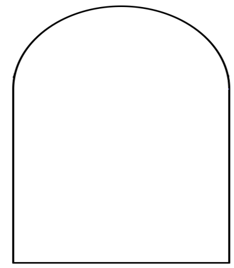
5. A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 50 centimeters by 80 centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?



6. Prove that for any real numbers a and b, if $a + b = 1$, then $a^4 + b^4 \geq \frac{1}{8}$.

7. One thousand feet of fencing is to be used to surround two areas, one square and one circular. What should the size of each area be in order that the total area be
   a) as large as possible b) as small as possible?

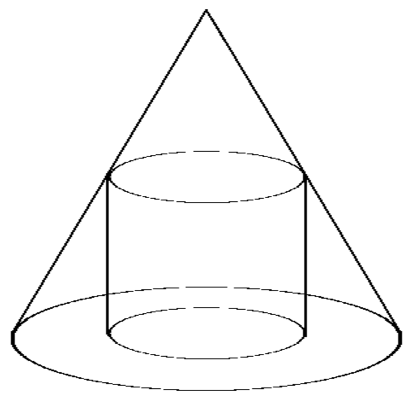
8. A Norman window has the outline of a semicircle on top of a rectangle, as shown on the picture. Find the dimensions of the window that can be built using 8 meters of wood and has the maximal area.



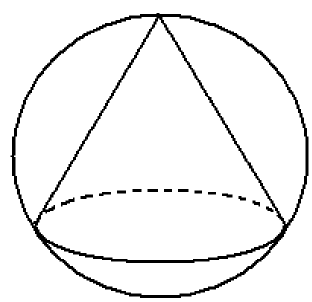
9. Find the point(s) on the arc of the parabola $y = x^2$ nearest to the point $(0, 2)$.

10. A company wants to manufacture cylindrical aluminum cans using $100\,\text{cm}^2$ of aluminum. What dimensions would guarantee the maximal volume?

11. We write a cylinder into a cone as shown on the picture. The cone is of height 15 cm and its base has a radius of 6 cm. What dimensions of the cylinder would guarantee the greatest volume?



12. Consider a spehere with radius R and all right cones we can write into it as shown on the picture. Which one has the greatest volume?

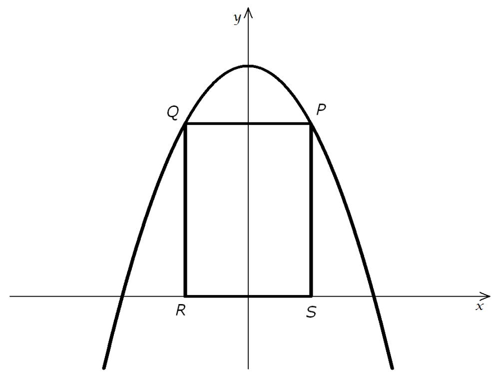


Practice Problems

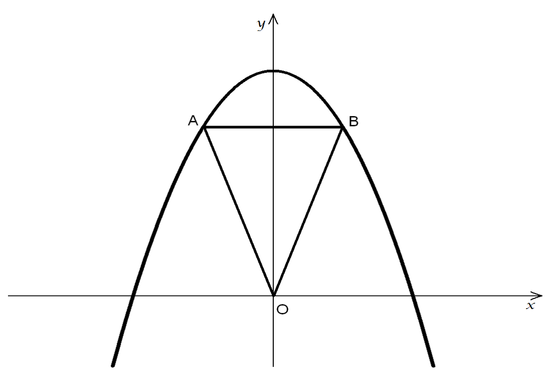
1. Consider $f(x) = x^3 + 3x^2 - 9x - 12$ on the interval $[-4, 4]$.
   a) Find all values of x for which f has a relative maximum.
   b) Find all values of x for which f has a relative minimum.
   c) Find all absolute maximums and minimums of f.

2. Consider $f(x) = x^3 - 3x^2 + 6$ on the interval $[-1, 4]$.
   a) Find all values of x for which f has a relative maximum.
   b) Find all values of x for which f has a relative minimum.
   c) Find all absolute maximums and minimums of f.

3. a) Let $P(x, y)$ be a point on the graph of $y = 4 - x^2$ with $0 \leq x \leq 2$. Let $PQRS$ be a rectangle with one side on the $x-$axis and two vertices on the graph, as shown on the picture below. Find the exact value of the coordinates of P and of the greatest possible area of such a rectangle.

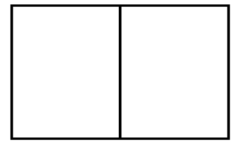


   b) Let $B(x, y)$ be a point on the graph of $y = 1 - x^2$ with $0 \leq x \leq 1$. Let ABO be an isosceles triangle with one horizontal side and third vertex at the origin, as shown on the picture below. Find the exact value of the coordinates of B and of the greatest possible area of such a triangle.



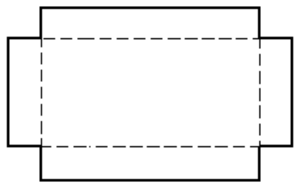
4. a) One positive number plus the square of another equals 15. Choose the numbers so that their product is as large as possible.
   b) Suppose that $S > 0$. One positive number plus the square of another equals S. Choose the numbers so that their product is as large as possible.

5. We have P meters of fencing and want to create two adjacent rectangular enclosings as shown on the figure below. What is the maximal area we can enclose this way? What dimensions will guarantee this maximal area?



6. a) A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 12 centimeters by 24 centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?

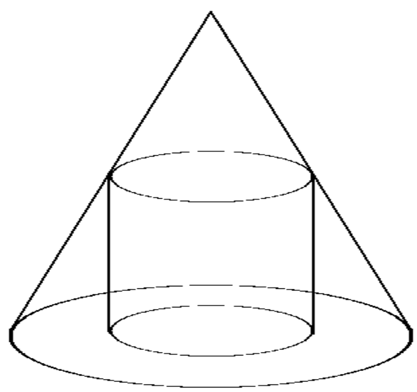
b) Suppose that $A, B > 0$. A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard A centimeters by B centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?



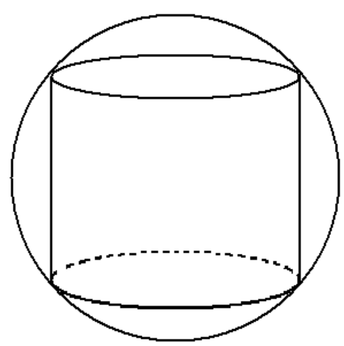
7. a) Find the point(s) on the line $3x + y = 12$ nearest to the origin.
   b) Find the point(s) on the line $3x + y = 12$ nearest to the point $(2, 10)$.

8. a) Find the point(s) on the graph of $y = x^2$ nearest to the point $(0, 3)$.
   b) Suppose that $B > 0$. Find the point(s) on the graph of $y = x^2$ nearest to the point $(0, B)$.

9. a) We write a cylinder into a cone as shown on the picture below. The cone is of height 24 cm and its base has a radius of 12 cm. What dimensions of the cylinder would guarantee the greatest volume?
   b) We write a cylinder into a cone as shown on the picture below. The cone is of height H and its base has a radius of R. What dimensions of the cylinder would guarantee the greatest volume?



10. Consider all right cylinders written into a sphere with radius R as shown on the picture below. What dimensions (height and radius) will guarantee the greatest volume for the cylinder?



Sample Problems - Answers

1.) a) $-\sqrt{2}$, $\sqrt{2}$ b) 0 c) absolute minimum: $(-3,-43)$ and $(3,-43)$; absolute maximum: $\left(-\sqrt{2},6\right)$ and $\left(\sqrt{2},6\right)$

2.) $\dfrac{4\sqrt{3}}{9}$ 3.) 4 and 32 4.) $\dfrac{P^2}{32}$ 5.) 10 cm by 10 cm 6.) see solutions

7.) a) $r_{\max} = \dfrac{500}{\pi}$ and $s_{\max} = 0$ b) $r_{\min} = \dfrac{500}{4+\pi}$ and $s_{\min} = \dfrac{1000}{\pi+4}$ 8.) width: $\dfrac{16}{\pi+4}$ height: $\dfrac{16}{\pi+4}$

9.) $\left(-\sqrt{\dfrac{3}{2}}, \dfrac{3}{2}\right)$ and $\left(\sqrt{\dfrac{3}{2}}, \dfrac{3}{2}\right)$ 10.) $r = \sqrt{\dfrac{50}{3\pi}}$ and $h = 2\sqrt{\dfrac{50}{3\pi}}$

11.) $V_{\max} = 80\pi\,\text{cm}^3$, when radius is 4 cm and height 5 cm 12.) $h = \dfrac{4}{3}R$ $r = \dfrac{2\sqrt{2}}{3}R$ and $V_{\max} = \dfrac{32}{81}\pi R^3$

Practice Problems - Answers

1.) a) -3 b) 1 c) absolute maximum: $(4,64)$ absolute minimum: $(1,-17)$

2.) a) 0 b) 2 c) absolute maximum: $(4,22)$ absolute minimum: $(-1,2)$ and $(2,2)$

3.) a) $P\left(\dfrac{2\sqrt{3}}{3}, \dfrac{8}{3}\right)$ Area $= \dfrac{32\sqrt{3}}{9}$ b) $B\left(\dfrac{\sqrt{3}}{3}, \dfrac{2}{3}\right)$ Area $= \dfrac{2\sqrt{3}}{9}$

4.) a) $\sqrt{5}, 10$ b) $\sqrt{\dfrac{S}{3}}, \dfrac{2}{3}S$ 5.) horizontal side: $\dfrac{P}{4}$ vertical side: $\dfrac{P}{6}$ Area: $\dfrac{P^2}{24}$

6.) a) $6+2\sqrt{3}$ b) $\dfrac{A+B+\sqrt{A^2-AB+B^2}}{6}$ 7.) a) $\left(\dfrac{18}{5}, \dfrac{6}{5}\right)$ b) $(4,0)$

8.) a) $\left(-\sqrt{\dfrac{5}{2}}, \dfrac{5}{2}\right)$ and $\left(\sqrt{\dfrac{5}{2}}, \dfrac{5}{2}\right)$

b) If $B \le \dfrac{1}{2}$, then the origin is closest. If $B > \dfrac{1}{2}$, then $\left(-\sqrt{B-\dfrac{1}{2}}, B-\dfrac{1}{2}\right)$ and $\left(\sqrt{B-\dfrac{1}{2}}, B-\dfrac{1}{2}\right)$

9.) a) 8 cm radius, 8 cm height, $V_{\max} = 512\pi\,\text{cm}^3$ b) $\dfrac{2}{3}R$ radius, $\dfrac{1}{3}H$ height, $V_{\max} = \dfrac{4}{27}\pi R^2 H$

10.) $h = \dfrac{2\sqrt{3}}{3}R$ and $r = \dfrac{\sqrt{6}}{3}R$ $V_{\max} = \dfrac{4\sqrt{3}}{9}\pi R^3$

Sample Problems - Solutions

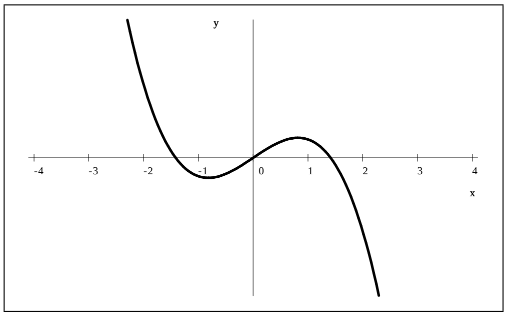
1.) Consider $f(x) = -x^4 + 4x^2 + 2$ on the interval $[-3, 3]$.

a) Find all values of x for which f has a relative maximum.

Solution: We differentiate f and sketch the derivative f'. All relative maximums will be where the derivative f' changes sign from positive to negative, provided that f is continuous there. Since f is a polynomial, it is continuous on its entire domain and so we will find the maximum where f' changes sign from positive to negative.

$$\begin{aligned} f(x) &= -x^4 + 4x^2 + 2 \\ f'(x) &= -4x^3 + 8x = -4x\left(x^2 - 2\right) = -4x\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right) \end{aligned}$$

We sketch the graph of the derivative.



Graph of $y = f'(x)$

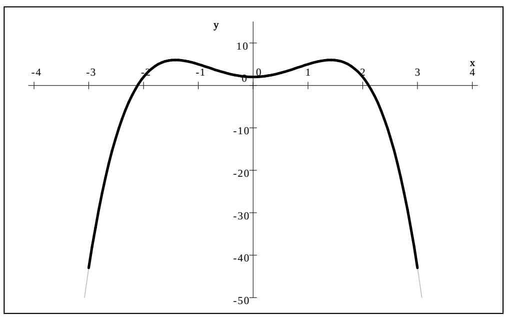
Since the derivative f' changes sign from positive to negative at $x = -\sqrt{2}$ and $x = \sqrt{2}$, f has relative maximums there.

b) Find all values of x for which f has a relative minimum.

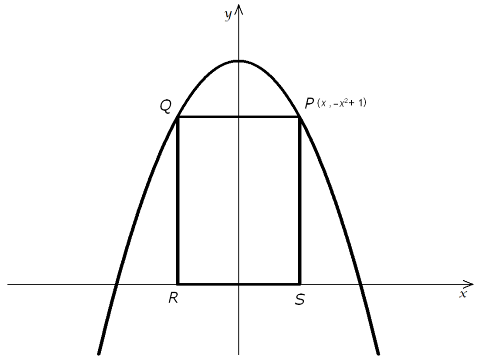
Solution: All relative minimums will be where the derivative f' changes sign from negative to positive, provided that f is continuous there. Since f is a polynomial, it is continuous on its entire domain and so we will find the relative minimum where f' changes sign from negative to positive. We sketched the graph of f' in part a) and so it is easy to see that f has a relative minimum at $x = 0$.

c) Find all absolute maximums and minimums of f.

Solution: If a function is continuous on a closed interval, it always achieves an absolute maximum and minimum. The candidates for these are the realtive extream and the endpoints of the domain. Thus, all we have to do is to evaluate f at $x = -3, -\sqrt{2}, 0, \sqrt{2}, 3$. (Note that if we notice that f is an even function, that can save us a few computations.) We compute that $f(-3) = -43$, $f\left(-\sqrt{2}\right) = 6$, $f(0) = 2$, $f\left(\sqrt{2}\right) = 6$, and $f(3) = -43$. Thus f has absolute minimums $(-3, -43)$ and $(3, -43)$ and absolute maximums $\left(-\sqrt{2}, 6\right)$ and $\left(\sqrt{2}, 6\right)$.



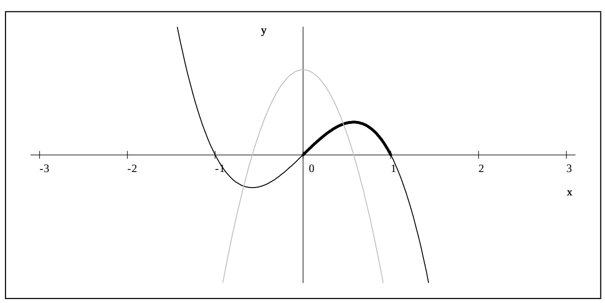
2.) Let $P(x,y)$ be a point on the graph of $y=-x^2+1$ with $0\le x\le 1$. Let $PQRS$ be a rectangle with one side on the $x-$ axis and two vertices on the graph, as shown on the picture below. Find the exact value of the greatest possible area of such a rectangle.



Solution: Let P be the point with coordinates $\left(x,-x^2+1\right)$. For the rectangle to have non-zero sides, $0<x<1$. Using the notation described in the problem, the horizontal side of the rectangle is $2x$ long, and the vertical side is $-x^2+1$ long. Thus the area of the rectangle, as a function of x, is $A(x)=2x\left(-x^2+1\right)=-2x^3+2x$, on domain $(0,1)$. The derivative is $A'(x)=-6x^2+2$.

$$A'(x)=-6x^2+2=-6\left(x^2-\frac{1}{3}\right)=-6\left(x+\frac{1}{\sqrt{3}}\right)\left(x-\frac{1}{\sqrt{3}}\right)$$

which is a downward opening parabola, with $x-$intercepts at $x=-\dfrac{1}{\sqrt{3}}$ and $x=\dfrac{1}{\sqrt{3}}$. Based on the sign of A', we determine that A has a relative minimum at $x=-\dfrac{1}{\sqrt{3}}$ and a relative maximum at $x=\dfrac{1}{\sqrt{3}}$. However, the domain of our function is $(0,1)$. If we sketch A on the domain $(0,1)$, it is clear from the graph that $A(x)$ has an absolute maximum at $x=\dfrac{1}{\sqrt{3}}$



Thus the greatest area is $A\left(\dfrac{1}{\sqrt{3}}\right)$. We compute this number:

$$A\left(\frac{1}{\sqrt{3}}\right)=2\left(\frac{1}{\sqrt{3}}\right)\left(-\left(\frac{1}{\sqrt{3}}\right)^2+1\right)=2\left(\frac{\sqrt{3}}{3}\right)\left(-\frac{1}{3}+1\right)=2\left(\frac{\sqrt{3}}{3}\right)\left(\frac{2}{3}\right)=\frac{4\sqrt{3}}{9}$$

And so the greatest possible area is $\dfrac{4\sqrt{3}}{9}$.

3.) One positive number plus the square of another equals 48. Choose the numbers so that their product is as large as possible.

Solution: Let us denote the numbers by a and b. We have $a+b^2=48 \implies a=48-b^2$. Then the product of the two numbers is a function of b,

$$P(b)=b\left(48-b^2\right)=-b^3+48b$$

Since this is a cubic function, it does not have an absolute maximum, only if its domain is restricted. The domain is

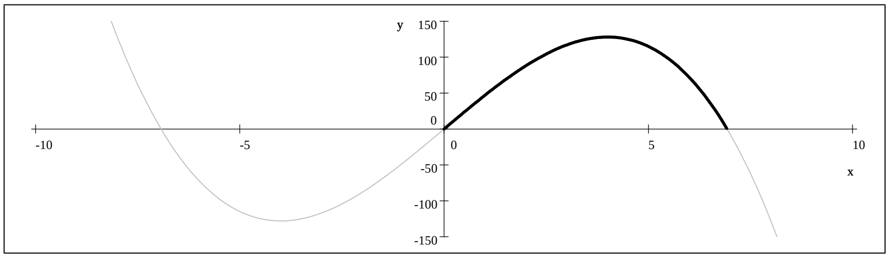
$$b \text{ is positive } \implies b>0 \quad \text{and}$$

$$a \text{ is positive } \implies 48-b^2>0 \quad \text{we solve this for } b \text{ and get}$$

$$\begin{aligned} 48-b^2 &> 0 \\ 48 &> b^2 \\ -\sqrt{48} &< b<\sqrt{48} \end{aligned}$$

which means that the domain of P is $\left(0, \sqrt{48}\right)$. (This is the set of all numbers for which both a and b are positive.) We can sketch P; it is a cubic polynomial with a negative leading coefficient, and its zeroes are at $x=0, -\sqrt{48}, \sqrt{48}$ as its factored form is

$$P(b)=-b\left(b^2-48\right)=-b\left(b+\sqrt{48}\right)\left(b-\sqrt{48}\right)$$

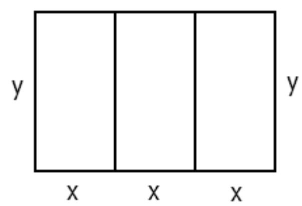


The picture shows that P will have a **relative** maximum on the interval $\left(0, \sqrt{48}\right)$ that is **also absolute maximum**. We find its $x-$coordinate by solving for the zero of the derivative.

$$\begin{aligned} P(b) &= -b^3+48b \\ P'(b) &= -3b^2+48=-3\left(b^2-16\right)=-3(b+4)(b-4) \end{aligned}$$

Thus the maximum is at $b=4$. Then the other number a is $48-b^2=48-16=32$.

4.) We have P meters of fencing and want to create three adjacent rectangular enclosings as shown on the figure below. What is the maximal area we can enclose this way?



Solution: Let x denote the small horizontal sides, and y the vertical sides.

$$\begin{aligned} P &= 6x+4y \qquad \implies y=\frac{1}{4}(P-6x)=-\frac{3}{2}x+\frac{P}{4} \\ A &= 3xy=3x\left(-\frac{3}{2}x+\frac{P}{4}\right)=-\frac{9}{2}x^2+\frac{3P}{4}x \end{aligned}$$

We compute the domain of this function. Both x and y must be positive. This gives us the following inequalities: $x>0$ and $y=-\frac{3}{2}x+\frac{P}{4}>0$

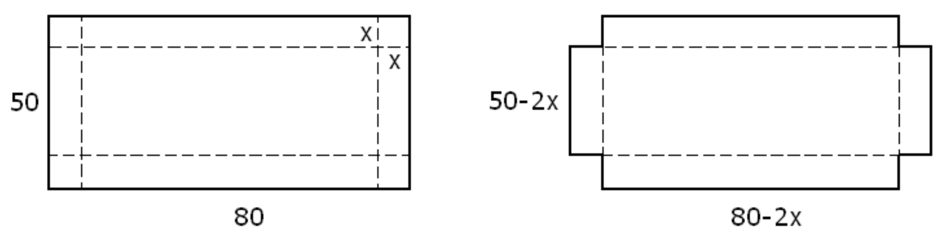
$$\begin{aligned} -\frac{3}{2}x+\frac{P}{4} &> 0 \qquad \text{solve for } x \\ \frac{P}{4} &> \frac{3}{2}x \\ \frac{P}{6} &> x \qquad \Longrightarrow \qquad \text{domain: } \left(0,\frac{P}{6}\right) \end{aligned}$$

The area function $A(x)=-\frac{9}{2}x^2+\frac{3P}{4}x$ is a downward turning parabola, so its vertex is its maximum. We can find it by completing the square or by differentiation:

$$A(x)=-\frac{9}{2}x^2+\frac{3P}{4}x=-\frac{9}{2}\left(x^2-\frac{P}{6}x\right)=-\frac{9}{2}\left(\left(x-\frac{P}{12}\right)^2-\frac{P^2}{144}\right)=-\frac{9}{2}\left(x-\frac{P}{12}\right)^2+\frac{P^2}{32}$$

When $x=\frac{P}{12}$, then the maximal area is $\frac{P^2}{32}$.

5.) A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 50 centimeters by 80 centimeters by cutting out equal squares in the corners and folding up the sides. What sides squares should be cut out for the container to have maximal volume?



Solution: The base of the box is a rectangle with sides $50-2x$ and $80-2x$. The height of the box is x. Thus the volume of the box, as a function of x is

$$V(x)=(80-2x)\,x\,(50-2x)=4\left(x^3-65x^2+1000x\right)$$

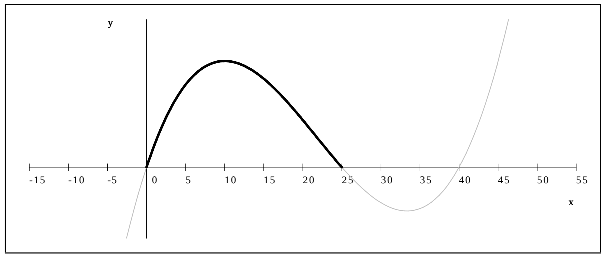
The domain is not the entire number line, clearly all sides must be positive. after we solve the inequalities $80-2x>0$ and $50-2x>0$ and $x>0$, we obtain the domain $(0,25)$. Since V is a cubic function with a positive leading coefficient, the relative extrema at the smaller $x-$value is a maximum, and the extrema at the greater $x-$value is a relative minimum. The domain is $(0,25)$.

$$V(x)=4\left(x^3-65x^2+1000x\right) \quad\Longrightarrow\quad V'(x)=4\left(3x^2-130x+1000\right)$$

We factor V' to find the zeroes of the derivative

$$\begin{aligned} V'(x) &= 4\left(3x^2-130x+1000\right)=4(3x-100)(x-10) \\ V'(x) &= 0 \quad\Longrightarrow\quad x_1=10 \qquad x_2=\frac{100}{3} \end{aligned}$$

The relative maximum is at $x=10$. This is an absolute maximum since the function is restricted to a domain of $(0,25)$.



So we should cut out squares with 10 centimeters long sides. Then the maximal volume is
$\times V(10) = (80 - 2 \cdot 10)(10)(50 - 2 \cdot 10) = 18\,000$.

6.) Prove that for any real numbers a and b, if $a + b = 1$, then $a^4 + b^4 \geq \frac{1}{8}$.

Solution: We will prove that the function $f(x) = x^4 + (1-x)^4$ has an absolute minimum at $x = \frac{1}{2}$.

$$\begin{aligned} f(x) &= x^4 + (1-x)^4 = x^4 + (x-1)^4 \\ f'(x) &= 4x^3 + 4(x-1)^3 = 4\left(x^3 + (x-1)^3\right) \end{aligned}$$

We factor via the sum of cubes theorem

$$\begin{aligned} f'(x) &= 4(x + x - 1)\left(x^2 - x(x-1) + (x-1)^2\right) = 4(2x-1)\left(x^2 - x + 1\right) \\ &= 8\left(x - \frac{1}{2}\right)\left(x^2 - x + 1\right) \end{aligned}$$

In case of the quadratic factor, we complete the square to see when that expression is negative.

$$8\left(x - \frac{1}{2}\right)\left(x^2 - x + 1\right) = 8\left(x - \frac{1}{2}\right)\left(\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}\right)$$

We can now see that the quadratic expression is always positive. (This is always the case with the quadratic factor in the sum or difference of two cubes.) Thus f' will be negative on $\left(-\infty, \frac{1}{2}\right)$ and positive on $\left(\frac{1}{2}, \infty\right)$. Thus f is strictly decreasing on $\left(-\infty, \frac{1}{2}\right)$ and strictly increasing on $\left(\frac{1}{2}, \infty\right)$. This means that f has an **absolute** minimum at $x = \frac{1}{2}$. We compute $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 + \left(1 - \frac{1}{2}\right)^4 = \frac{1}{8}$. (Note: this problem can be solved by elementary methods as well. One method involves simplifying $\left(\frac{1}{2} + x\right)^4 + \left(\frac{1}{2} - x\right)^4$ and completing the square.)

7.) One thousand feet of fencing is to be used to surround two areas, one square and one circular. What should the size of each area be in order that the total area be
a) as large as possible b) as small as possible?

Solution: Let us denote the radius of the circle by r. Clearly $r \geq 0$. If we denote the side of the square by s, we have an equation for the total perimeter. We can solve that for s.

$$\begin{aligned} 1000 &= 4s + 2\pi r \\ \frac{100 - 2\pi r}{4} &= s \qquad \Longrightarrow \qquad s = \frac{500 - \pi r}{2} \end{aligned}$$

We also need $s \geq 0$

$$\begin{aligned} s &\geq 0 & 500 &\geq \pi r \\ \frac{500 - \pi r}{2} &\geq 0 & \frac{500}{\pi} &\geq r \\ 500 - \pi r &\geq 0 \end{aligned}$$

Thus the area-function, (yet to be set up) will have domain $\left[0, \dfrac{500}{\pi}\right]$. The endpoints mean that all 1000 feet was used for the square (in case of $r=0$) or for the circle (in case of $r=\dfrac{500}{\pi}$). The area, as a function of r is now

$$A(r)=\pi r^2+s^2=\pi r^2+\left(\frac{500-\pi r}{2}\right)^2=\left(\pi+\frac{1}{4}\pi^2\right)r^2-250\pi r+62\,500$$

Since this is a quadratic expression with a positive leading coefficient, we know that the vertex is a minimum, both relative and absolute. The vertex can be found by either elementary methods or by finding the zero of $A'(r)$. Either way, we will get that $r_{\min}=\dfrac{500}{4+\pi}$. This means that the circle will have radius $r=\dfrac{500}{4+\pi}$ and the square will have sides $s_{\min}=\dfrac{500-\pi r_{\min}}{2}=\dfrac{1000}{\pi+4}$ since

$$\begin{aligned} s_{\min} &= \frac{500-\pi r_{\min}}{2}=\frac{500-\pi\left(\frac{500}{4+\pi}\right)}{2}=\frac{500-\pi\left(\frac{500}{4+\pi}\right)}{2}\cdot\frac{4+\pi}{4+\pi}=\frac{500(4+\pi)-500\pi}{2(4+\pi)} \\ &= \frac{2000+500\pi-500\pi}{2(4+\pi)}=\frac{2000}{2(4+\pi)}=\frac{1000}{\pi+4}=2r_{\min} \end{aligned}$$

These dimensions give us the minimal area

$$\begin{aligned} A_{\min} &= \pi r_{\min}^2+s_{\min}^2=\pi\left(\frac{500}{4+\pi}\right)^2+\left(\frac{1000}{\pi+4}\right)^2=\frac{500^2\pi+1000^2}{(4+\pi)^2}=\frac{500^2\pi+(2\cdot 500)^2}{(4+\pi)^2} \\ &= \frac{500^2\pi+4\cdot 500^2}{(4+\pi)^2}=\frac{500^2(\pi+4)}{(4+\pi)^2}=\frac{500^2}{(4+\pi)}=\frac{250\,000}{\pi+4} \end{aligned}$$

But what about the maximum? A quadratic expression does not have a relative maximum if its leading coefficient is positive. The absolute maximum must be at one of the end-points of the interval $\left[0, \dfrac{500}{\pi}\right]$. We need to compare these areas. When $r=0$, all thousand feet was used for the square. The area is then

$$A(0)=250^2$$

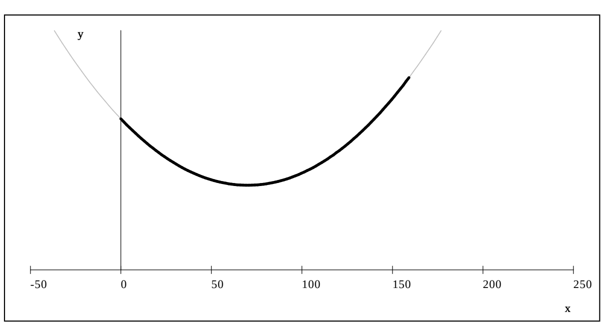
or, using our expression for the area,

$$A(r)=\left(\pi+\frac{1}{4}\pi^2\right)r^2-250\pi r+62\,500 \quad\Longrightarrow A(0)=62\,500$$

When $r=\dfrac{500}{\pi}$, all thousand feet was used for the circle. Then the area is

$$A\left(\frac{500}{\pi}\right)=\pi r^2=\pi\left(\frac{500}{\pi}\right)^2=\frac{500^2}{\pi}$$

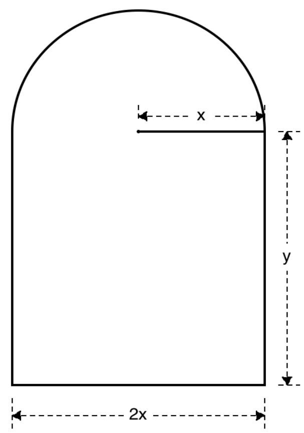
Since $\dfrac{500^2}{\pi}\approx 79\,577.471\,545\,5$



$$A(0) < A\left(\frac{500}{\pi}\right)$$

which means that the greatest area occurs if we use all thousand feet for the circle.

8.) A Norman window has the outline of a semicircle on top of a rectangle, as shown on the picture below. Find the dimensions of the window that can be built using 8 meters of wood and has the maximal area.



Solution: Let x denote half of the bottom side. (Also the radius.)
Let y denote the vertical side. Then $2x + 2y + \pi x = 8$. We solve for y and obtain

$$y = -\left(\frac{\pi+2}{2}\right)x + 4$$

The domain will be determined by the conditions $x > 0$ and $y \geq 0$. The domain is $\left(0, \frac{8}{\pi+2}\right]$.We set up the area-function:

$$A(x) = 2xy + \frac{\pi x^2}{2} = 2x\left(-\left(\frac{\pi+2}{2}\right)x + 4\right) + \frac{\pi x^2}{2} = \left(-\frac{\pi}{2} - 2\right)x^2 + 8x$$

The vertex of this downward opening parabola is at $x = \frac{8}{\pi+4}$, which is in the domain. Then $y = \frac{8}{\pi+4}$. The dimensions are: $2x = \frac{16}{\pi+4}$ wide and $x + y = \frac{16}{\pi+4}$ tall.

9.) Find the point(s) on the arc of the parabola $y = x^2$ that are nearest to the point $(0, 2)$.

Solution: Let $P\left(x, x^2\right)$ be a point on the graph. The distance between P and $(0, 2)$ is

$$d = \sqrt{(x-0)^2 + (x^2-2)^2} = \sqrt{x^2 + x^4 - 4x^2 + 4} = \sqrt{x^4 - 3x^2 + 4}$$

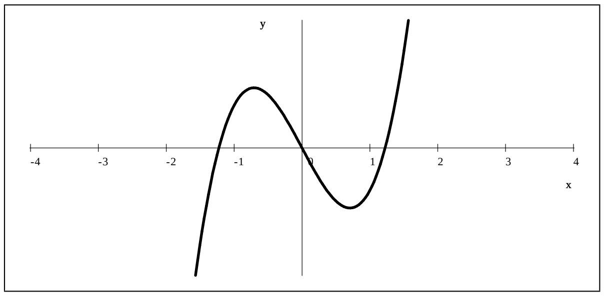
It is a common trick to find the minimum of d^2 instead of d. Let $f(x)$ denote the square of the distance between the point $\left(x, x^2\right)$ and $(0, 2)$.

$$f(x) = x^4 - 3x^2 + 4$$

We differentiate:

$$f'(x) = 4x^3 - 6x = 4x\left(x^2 - \frac{3}{2}\right) = 4x\left(x + \sqrt{\frac{3}{2}}\right)\left(x - \sqrt{\frac{3}{2}}\right)$$

Since f' is a cubic function with a positive leading coefficient, and $x-$intercepts at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$. We sketch the graph of f'.

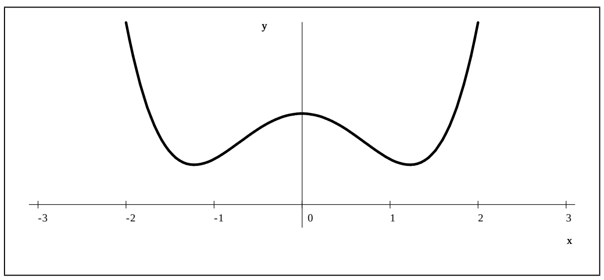


Graph of f'

Based on the signs of f', we determine that f has a relative minimum at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$ and a relative maximum at $x = 0$. We now evaluate f at $x = -\sqrt{\frac{3}{2}}, 0, \sqrt{\frac{3}{2}}$.

$$f(0) = 4, \quad f\left(-\sqrt{\frac{3}{2}}\right) = \frac{7}{4}, \quad f\left(\sqrt{\frac{3}{2}}\right) = \frac{7}{4}$$

We can now plot the graph of f.



Graph of f

The absolute minimum is at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$ indicating that the points closest to $(0,2)$ are $\left(-\sqrt{\frac{3}{2}}, \frac{3}{2}\right)$ and $\left(\sqrt{\frac{3}{2}}, \frac{3}{2}\right)$.

10.) A company wants to manufacture cylindrical aluminum cans using 100 cm^2 of aluminum. What dimensions would guarantee the maximal volume?

Solution: Let h denote the height of the can, and r denote the radius of the base circle. The surface of a cylinder is $A = 2\pi r^2 + 2\pi rh$. We solve for h.

$$100 = 2\pi r^2 + 2\pi rh \quad \Longrightarrow \quad h = \frac{100 - 2\pi r^2}{2\pi r} = -r + \frac{50}{\pi r}$$

The volume of a cylinder is $V = \pi r^2 h$. We substitute $h = \dfrac{100 - 2\pi r^2}{2\pi r}$ into this equation and obtain V as a function of r.

$$\begin{aligned} V(r) &= \pi r^2 \left(\frac{100 - 2\pi r^2}{2\pi r}\right) = r\left(\frac{100 - 2\pi r^2}{2}\right) = r\left(50 - \pi r^2\right) \\ V(r) &= -\pi r^3 + 50r \end{aligned}$$

The volume is a cubic function with a negative leading coefficient. We can also determine the domain of V by stating that both r and h must be positive.

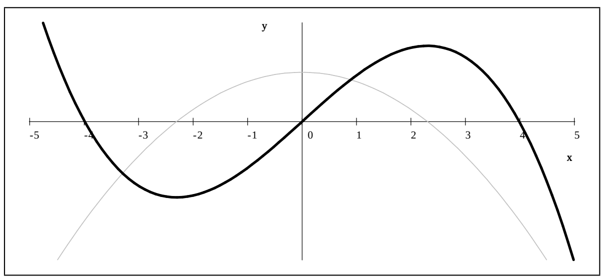
$$\begin{aligned} h &> 0 \text{ and } r > 0 \\ \frac{100-2\pi r^2}{2\pi r} &> 0 \qquad \text{multiply by } 2\pi r > 0 \\ 100-2\pi r^2 &> 0 \\ 100 &> 2\pi r^2 \\ \frac{50}{\pi} &> r^2 \implies -\sqrt{\frac{50}{\pi}} < r < \sqrt{\frac{50}{\pi}} \qquad \text{and } \; r > 0 \end{aligned}$$

The domain of the volume function is thus $\left(0, \sqrt{\frac{50}{\pi}}\right)$.

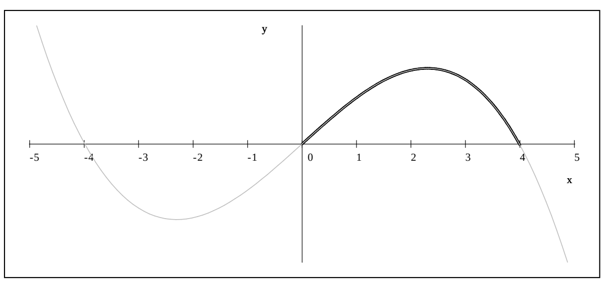
We differentiate $V(r)$.

$$V'(r) = -3\pi r^2 + 50 = -3\pi\left(r^2 - \frac{50}{3\pi}\right) = -3\pi\left(r + \sqrt{\frac{50}{3\pi}}\right)\left(r - \sqrt{\frac{50}{3\pi}}\right)$$

The derivative is a downward turned parabola with $x-$intercepts at $-\sqrt{\frac{50}{3\pi}}$ and $\sqrt{\frac{50}{3\pi}}$. The positive one, $r = \sqrt{\frac{50}{3\pi}}$ is where the volume has a relative maximum. We can now sketch the volume function V.



If we restrict the function to its domain, $\left(0, \sqrt{\frac{50}{\pi}}\right)$, then it is clear that the relative maximum we found is also an absolute maximum.

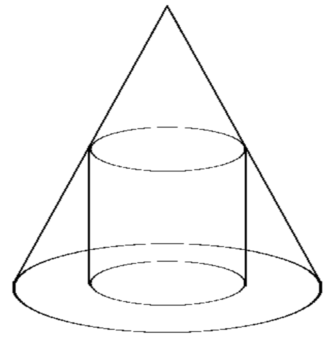


So the maximal volume will be obtained with $r = \sqrt{\frac{50}{3\pi}}$. Then $h = -r + \frac{50}{\pi r}$

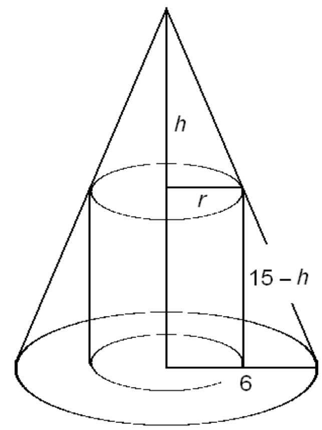
$$\begin{aligned} h &= -\sqrt{\frac{50}{3\pi}} + \frac{50}{\pi\sqrt{\frac{50}{3\pi}}} = -\sqrt{\frac{50}{3\pi}} + \frac{50}{\pi}\sqrt{\frac{3\pi}{50}} = -\sqrt{\frac{50}{3\pi}} + \sqrt{\frac{50^2}{\pi^2}\cdot\frac{3\pi}{50}} = \\ &= -\sqrt{\frac{50}{3\pi}} + \sqrt{9\cdot\frac{50}{3\pi}} = -\sqrt{\frac{50}{3\pi}} + \sqrt{9}\sqrt{\frac{50}{3\pi}} = -\sqrt{\frac{50}{3\pi}} + 3\sqrt{\frac{50}{3\pi}} = 2\sqrt{\frac{50}{3\pi}} = 2r \end{aligned}$$

and so the maximal volume is $V\left(\sqrt{\frac{50}{3\pi}}\right)$.

11.) We write a cylinder into a cone as shown on the picture. The cone is of height 15 cm and its base has a radius of 6 cm. What dimensions of the cylinder would guarantee the greatest volume? What is the maximal volume?



Solution: The cone above the cylinder and the big cone are similar to each other. Let us denote the height of this cone by h and the radius of its base by r. Then the cylinder's height is $15 - h$ and its base has radius r.



Since the two cones are similar, $\dfrac{15}{6} = \dfrac{h}{r}$. We solve for h and obtain $h = \dfrac{5}{2}r$. We can now express the volume of the cylinder as a function of r.

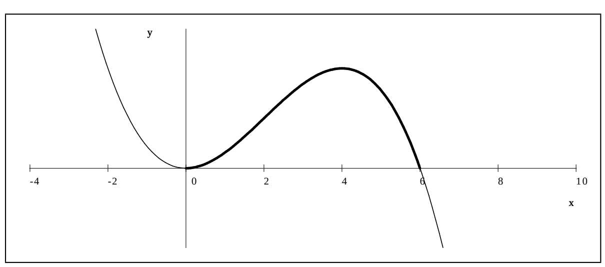
$$V_{\text{cylinder}} = \pi r^2 (15 - h) = \pi r^2 \left(15 - \frac{5}{2}r\right) = \pi \left(-\frac{5}{2}r^3 + 15r^2\right)$$

It is also clear that $0 < r < 6$ and so the volume of the cylinder as a function of r is

$$V(r) = \pi \left(-\frac{5}{2}r^3 + 15r^2\right) \quad \text{on domain } (0, 6)$$

$V(r)$ is clearly a cubic function with a negative leading coefficient and zeroes at 0 and r, since

$$V(r) = \pi r^2 \left(15 - \frac{5}{2}r\right) = \pi r^2 \left(-\frac{5}{2}\right)(r - 6) = -\frac{5}{2}\pi r^2 (r - 6)$$



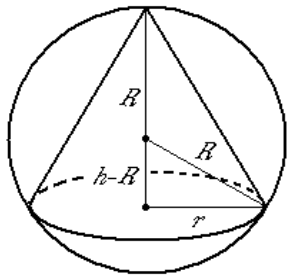
We can see from the graph that there is a relative minimum at 0 and a relative maximum between 0 and 6 that is an absolute maximum if the domain is $(0, 6)$. To find this maximum, we differentiate $V(r)$.

$$\begin{aligned} V(r) &= \pi \left(-\frac{5}{2}r^3 + 15r^2\right) \\ V'(r) &= \pi \left(-\frac{15}{2}r^2 + 30r\right) = -\frac{15}{2}\pi r (r - 4) \end{aligned}$$

The derivative has zeroes at $r=0$ and $r=4$. Thus the maximum volume is achieved when $r=4$. Then clearly $h=\frac{5}{2}r=\frac{5}{2}\cdot 4=10$. This is not the height of the cylinder; that is $15-h=15-10=5$. Dimensions are in centimeters and so the maximal volume is $V_{\max}=\pi\left(4\,\mathrm{cm}\right)^2\left(5\,\mathrm{cm}\right)=80\pi\,\mathrm{cm}^3$ when the radius of the cylinder is 4 cm and its height is 5 cm.

12.) Consider a spehere with radius R and all right cones we can write into it. Which one has the greatest volume?

Solution: Let us denote the height of the cone by h and the base of its radius by r. Consider now the picture shown below.



We can write the Pythagorean Theorem on the right tringle and get $R^2=(h-R)^2+r^2$ from which we solve for r^2 and get that

$$r^2=R^2-(h-R)^2=R^2-\left(R^2+h^2-2hR\right)=2hR-h^2$$

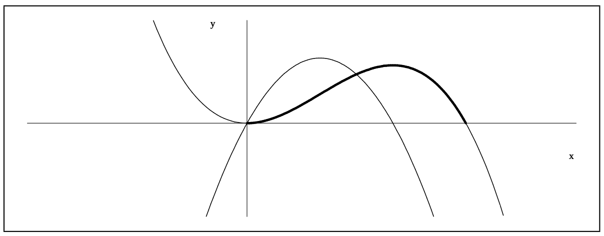
Now the volume of the cone is $V=\frac{1}{3}\pi r^2 h$. We substitute $r^2=2hR-h^2$ into this and the volme becomes a function of h.

$$V(h)=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(2hR-h^2\right)h=\frac{\pi}{3}\left(-h^3+2h^2R\right)$$

This is a cubic function with a negative leading coefficient with a domain $(R,2R)$. We differentiate V and factor V' to find its zeroes.

$$V'(h)=\frac{\pi}{3}\left(-3h^2+4hR\right)=\frac{\pi}{3}(-3)\left(h^2-\frac{4hR}{3}\right)=-\pi h\left(h-\frac{4R}{3}\right)$$

V' is a downward opening parabola with zeroes a $h=0$ and $h=\frac{4R}{3}$.



The greater one represents a relative maximum for f. On the restricted domain $(R,2R)$, V has an absolute maximum at $h=\frac{4}{3}R$ so it is our solution. Then

$$r=\sqrt{2hR-h^2}=\sqrt{2\left(\frac{4}{3}R\right)R-\left(\frac{4}{3}R\right)^2}=\sqrt{\frac{8}{3}R^2-\frac{16}{9}R^2}=R\sqrt{\frac{8}{3}-\frac{16}{9}}=R\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}R$$

and

$$V_{\max}=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\sqrt{\frac{8}{9}}R\right)^2\left(\frac{4}{3}R\right)=\frac{\pi}{3}\cdot\frac{8}{9}\cdot\frac{4}{3}R^3=\frac{32}{81}\pi R^3$$

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