Solutions of selected problems

4. Assume that for all real numbers x and y,

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1\\ \sin(x+y) &= \sin x\cos y + \cos x \sin y \quad \text{and}\\ \cos(x+y) &= \cos x \cos y - \sin x \sin y\end{aligned}$$

Prove each of the following.

a) $\sin(x-y) = \sin x\cos y - \cos x\sin y$
Solution: We will think of $x-y$ as $x+(-y)$ and apply the sum formula to this sum.

$$\sin(x-y) = \sin(x+(-y)) = \sin x\cos(-y) + \cos x\sin(-y)$$

We know that $\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$.

$$\sin(x-y) = \sin x\cos y + (-1)\cos x\sin y = \sin x\cos y - \cos x\sin y$$

b) $\cos 2x = 2\cos^2 x - 1$
Solution:

$$\cos 2x = \cos(x+x) = \cos x\cos x - \sin x\sin x = \cos^2 x - \sin^2 x$$

We can eliminate $\sin x$ using the identity $\cos^2 x + \sin^2 x = 1$; by substituting $\sin^2 x = 1-\cos^2 x$

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - \left(1-\cos^2 x\right) = 2\cos^2 x - 1$$

c) $\cos 2x = 1 - 2\sin^2 x$
We start with $\cos 2x = \cos^2 x - \sin^2 x$ and eliminate $\cos x$ using the identity $\sin^2 x + \cos^2 x = 1$; by substituting $\cos^2 x = 1-\sin^2 x$

$$\cos 2x = \cos^2 x - \sin^2 x = \left(1-\sin^2 x\right) - \sin^2 x = 1 - 2\sin^2 x$$

d) $\sin x = \pm\sqrt{\dfrac{1-\cos 2x}{2}}$
Solution: We start with $\cos 2x = 1-2\sin^2 x$ and solve for $\sin x$.

e) $\tan(x+y) = \dfrac{\tan x + \tan y}{1-\tan x\tan y}$
Solution:

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y}$$

We will now divide both numerator and denominator by $\cos x\cos y$

$$\begin{aligned}\tan(x+y) &= \frac{\dfrac{\sin x\cos y + \cos x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y - \sin x\sin y}{\cos x\cos y}} = \frac{\dfrac{\sin x\cos y}{\cos x\cos y} + \dfrac{\cos x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y}{\cos x\cos y} - \dfrac{\sin x\sin y}{\cos x\cos y}} = \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x}{\cos x}\dfrac{\sin y}{\cos y}}\\ &= \frac{\tan x + \tan y}{1-\tan x\tan y}\end{aligned}$$

f) $\sec^2 x = 1 + \tan^2 x$
Solution:

$$\text{RHS} = 1+\tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x = \text{LHS}$$

g) $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$
Solution: Start with $\cos 2x = 2\cos^2 x - 1$ and solve for $\cos^2 x$.

h) $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$
Solution:

$$\text{LHS} = \frac{\sin x}{1+\cos x} = \frac{\sin x}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x} = \frac{\sin x(1-\cos x)}{1-\cos^2 x} = \frac{\sin x(1-\cos x)}{\sin^2 x} = \frac{1-\cos x}{\sin x} = \text{RHS}$$

5. Simplify each of the following.

a) $\sin\left(\sin^{-1} x\right)$
Solution: When we compose a function with its inverse, we always get $f\left(f^{-1}(x)\right) = x$.

b) $\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$
Solution 1: We need to simplify $\cos\left(\sin^{-1} x\right)$. Let $\beta = \sin^{-1} x$. We re-write the expression to be simplified:

$$\cos\underbrace{\left(\sin^{-1} x\right)}_{\beta} = \ ?$$

This means that $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, $\sin\beta = x$, and we need to express $\cos\beta$ in terms of x. Since $\sin^2\beta + \cos^2\beta = 1$,

$$\cos\beta = \pm\sqrt{1-\sin^2\beta} = \pm\sqrt{1-x^2}$$

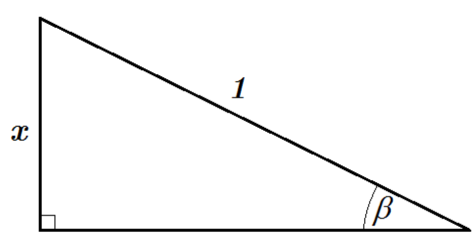
Since $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, $\cos\beta$ is positive and so $\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$

Solution 2: This method significantly reduces computation but can only provide us with the answer up to sign! **We always have to worry about the signs after the method gave us the absolute value of the answer.**

We need to simplify $\cos\left(\sin^{-1} x\right)$. Let $\beta = \sin^{-1} x$. We re-write the expression to be simplified:

$$\cos\underbrace{\left(\sin^{-1} x\right)}_{\beta} = \ ?$$

This means that $\sin\beta = x$, and we need to express $\cos\beta$ in terms of x. Let us first draw a right triangle where $\sin\beta = x$ happens. One side will be labeled as x, another as 1, and one angle as β so that it is true that $\sin\beta = x$. Here is such a triangle:



We compute the missing side by the Pythagorean theorem and get $\sqrt{1-x^2}$. Now we can easily compute $\cos\beta$ in terms of the triangle: $\cos\beta = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. This is a very nice easy method, but the payback is that we need to worry about the sign of the answer: so far we only know that it is $\pm\sqrt{1-x^2}$. We know that the range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The cosine of all angles within this interval is non-negative, so the answer is simply $\sqrt{1-x^2}$.

c) $\sin\left(\tan^{-1} x\right) = \dfrac{x}{\sqrt{x^2+1}}$

Solution 1: We need to simplify $\sin\left(\tan^{-1} x\right)$. Let $\gamma = \tan^{-1} x$. We re-write the expression to be simplified:

$$\sin \underbrace{\left(\tan^{-1} x\right)}_{\gamma} = ?$$

This means that $-\dfrac{\pi}{2} < \gamma < \dfrac{\pi}{2}$, $\tan\gamma = x$, and we need to express $\sin\gamma$ in terms of x.

$$\begin{aligned}
\sin^2\gamma + \cos^2\gamma &= 1 && \text{divide by } \sin^2\gamma \\
\frac{\sin^2\gamma}{\sin^2\gamma} + \frac{\cos^2\gamma}{\sin^2\gamma} &= \frac{1}{\sin^2\gamma} \\
1 + \frac{1}{\tan^2\gamma} &= \frac{1}{\sin^2\gamma} \\
\frac{\tan^2\gamma + 1}{\tan^2\gamma} &= \frac{1}{\sin^2\gamma} && \text{take reciprocal} \\
\frac{\tan^2\gamma}{\tan^2\gamma + 1} &= \sin^2\gamma \\
\sin\gamma &= \pm\sqrt{\frac{\tan^2\gamma}{1+\tan^2\gamma}} = \pm\frac{\sqrt{\tan^2\gamma}}{\sqrt{1+\tan^2\gamma}} = \pm\frac{|\tan\gamma|}{\sqrt{1+\tan^2\gamma}} = \pm\frac{|x|}{\sqrt{1+x^2}}
\end{aligned}$$

We can further simplify this expression by considering its sign. Since $-\dfrac{\pi}{2} < \gamma < \dfrac{\pi}{2}$, the value of $\cos\gamma$ is always non-negative. This means that $\sin\gamma$ and $\tan\gamma$ have the same signs, if one is positive, so is the other; if one is negative, so is the other. This means that our expression can be simplified as

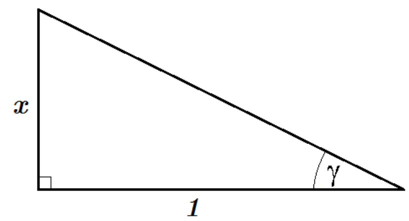
$$\sin\gamma = \frac{x}{\sqrt{1+x^2}}$$

Solution 2: This method significantly reduces computation but can only provide us with the answer up to sign! **We always have to worry about the signs after the method gave us the absolute value of the answer.**

We need to simplify $\sin\left(\tan^{-1} x\right)$. Let $\gamma = \tan^{-1} x$. We re-write the expression to be simplified:

$$\sin \underbrace{\left(\tan^{-1} x\right)}_{\gamma} = ?$$

This means that $\tan\gamma = x$, and we need to express $\sin\gamma$ in terms of x. Let us first draw a right triangle where $\tan\gamma = x$ happens. One side will be labeled as x, another as 1, and one angle as γ so that it is true that $\tan\gamma = x$. Here is such a triangle:



We compute the missing side by the Pythagorean theorem and get $\sqrt{x^2+1}$. Now we can easily compute $\sin\gamma$ in terms of the triangle: $\sin\gamma = \dfrac{x}{\sqrt{x^2+1}}$. This is a very nice easy method, but the payback is that we need to worry about the sign of the answer: so far we only know that it is $\pm\dfrac{x}{\sqrt{x^2+1}}$. We know that the range of $\tan^{-1}x$ is $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$. The cosine of all angles within this interval is non-negative, so the sign of tangent depends on the sign of sine. If the sine is postive, so is the tangent. If the sine is negative, so is the tangent. In the expression, $\dfrac{x}{\sqrt{x^2+1}}$, the denominator is clearly positive. The fraction is positive if x is, and negative if x is. Since $x=\tan\gamma$ and $\sin\gamma$ must have the same sign, our expression can be simplified as $\dfrac{x}{\sqrt{x^2+1}}$.

d) $\tan\left(\cos^{-1}x\right)=\dfrac{\sqrt{1-x^2}}{x}$

Solution 1: We need to simplify $\tan\left(\cos^{-1}x\right)$. Let $\theta=\cos^{-1}x$. This means that $0<\theta<\pi$, $\cos\theta=x$, and we need to express $\tan\theta$ in terms of x.

We can quickly figure out $\sin\theta$ (see part b): $\sin\theta=\pm\sqrt{1-\cos^2\theta}=\pm\sqrt{1-x^2}$. because $0<\theta<\pi$, the value of $\sin\theta$ is positive and so $\sin\theta=\sqrt{1-x^2}$ Then $\tan\theta=\dfrac{\sin\theta}{\cos\theta}=\dfrac{\sqrt{1-x^2}}{x}$

Solution 2:

$$\begin{aligned}
\sin^2\theta+\cos^2\theta &= 1 \qquad \text{divide by } \cos^2\theta\\
\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta}\\
\tan^2\theta+1 &= \frac{1}{\cos^2\theta}\\
\tan\theta &= \pm\sqrt{\frac{1}{\cos^2\theta}-1}=\pm\sqrt{\frac{1}{x^2}-1}=\pm\sqrt{\frac{1-x^2}{x^2}}=\pm\frac{\sqrt{1-x^2}}{\sqrt{x^2}}\\
\tan\theta &= \pm\frac{\sqrt{1-x^2}}{|x|}
\end{aligned}$$

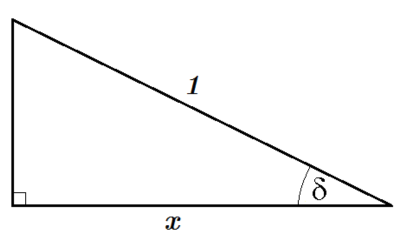
Since $0<\theta<\pi$, the sign of $\tan\theta$ will depend on the sign of $\cos\theta$ (clearly, $\sin\theta$ is positive on $(0,\pi)$. Thus $\tan\theta$ is positive when $\cos\theta=x$ is positive and negative when $\cos\theta=x$ is negative. This means that our expression can be simplified as $\tan\theta=\dfrac{\sqrt{1-x^2}}{x}$

Solution 3: This method significantly reduces computation but can only provide us with the answer up to sign! **We always have to worry about the signs after the method gave us the absolute value of the answer.**

We need to simplify $\tan\left(\cos^{-1}x\right)$. Let $\delta=\tan^{-1}x$. We re-write the expression to be simplified:

$$\tan\underbrace{\left(\cos^{-1}x\right)}_{\delta} = ?$$

This means that $\cos\delta=x$, and we need to express $\tan\delta$ in terms of x. Let us first draw a right triangle where $\cos\delta=x$ happens. One side will be labeled as x, another as 1, and one angle as δ so that it is true that $\cos\delta=x$. Here is such a triangle:



We compute the missing side by the Pythagorean theorem and get $\sqrt{1-x^2}$. Now we can easily compute $\tan\delta$ in terms of the triangle: $\tan\delta = \dfrac{\sqrt{1-x^2}}{x}$. This is a very nice easy method, but the payback is that we need to worry about the sign of the answer: so far we only know that it is $\pm\dfrac{\sqrt{1-x^2}}{x}$. Since $0<\theta<\pi$, the sign of $\tan\theta$ will depend on the sign of $\cos\theta$ (clearly, $\sin\theta$ is positive on $(0,\pi)$. Thus $\tan\theta$ is positive when $\cos\theta = x$ is positive and negative when $\cos\theta = x$ is negative. This means that our expression can be simplified as $\tan\theta = \dfrac{\sqrt{1-x^2}}{x}$.

6. Claim: $A = B = C = D = E$ where

$$A = \sqrt{\frac{1+\sin x}{1-\sin x}} \qquad B = \frac{1+\sin x}{\cos x} \qquad C = \sec x + \tan x \qquad D = \frac{\cos x}{1-\sin x} \qquad E = \frac{1}{\sec x - \tan x}$$

Proof:

$$A = \sqrt{\frac{1+\sin x}{1-\sin x}} = \sqrt{\frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}} = \sqrt{\frac{(1+\sin x)^2}{1-\sin^2 x}} = \sqrt{\frac{(1+\sin x)^2}{\cos^2 x}} = \sqrt{\left(\frac{1+\sin x}{\cos x}\right)^2} = \frac{1+\sin x}{\cos x} = B$$

$$B = \frac{1+\sin x}{\cos x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \sec x + \tan x = C$$

$$B = \frac{1+\sin x}{\cos x} = \frac{1+\sin x}{\cos x}\cdot\frac{1-\sin x}{1-\sin x} = \frac{1-\sin^2 x}{\cos x\,(1-\sin x)} = \frac{\cos^2 x}{\cos x\,(1-\sin x)} = \frac{\cos x}{1-\sin x} = D$$

$$D = \frac{\cos x}{1-\sin x} = \frac{1}{\dfrac{1-\sin x}{\cos x}} = \frac{1}{\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}} = \frac{1}{\sec x - \tan x} = E$$

7. a) $\log_{24} 90 = \dfrac{\ln 90}{\ln 24} = \dfrac{\ln\left(2\cdot 3^2\cdot 5\right)}{\ln\left(2^3\cdot 3\right)} = \dfrac{\ln 2 + \ln\left(3^2\right) + \ln 5}{\ln\left(2^3\right) + \ln 3} = \dfrac{\ln 2 + 2\ln 3 + \ln 5}{3\ln 2 + \ln 3}$

b) $2\log_{10}(2x) + \log_{10}(25x) - 3\log_{10} 0.1x = \log_{10}\dfrac{4x^2(25x)}{(0.1x)^3} = \log_{10}\left(\dfrac{100x^3}{\frac{1}{1000}x^3}\right) = \log_{10} 10^5 = 5$

c) $\log_2 3\cdot\log_3 4\cdot\log_4 5\cdot\log_5 6\cdot\log_6 7\cdot\log_7 8 = \dfrac{\ln 3}{\ln 2}\cdot\dfrac{\ln 4}{\ln 3}\cdot\dfrac{\ln 5}{\ln 4}\cdot\dfrac{\ln 6}{\ln 5}\cdot\dfrac{\ln 7}{\ln 6}\cdot\dfrac{\ln 8}{\ln 7} = \dfrac{\ln 8}{\ln 2} = \dfrac{\ln\left(2^3\right)}{\ln 2} = \dfrac{3\ln 2}{\ln 2} = 3$

d) $\log_3|\tan x| = \log_3\left|\dfrac{\sin x}{\cos x}\right| = \log_3\left|\left(\dfrac{\cos x}{\sin x}\right)^{-1}\right| = -\log_3\left|\dfrac{\cos x}{\sin x}\right| = -\log_3|\cot x|$

10. Claim: $\dfrac{d\left(x^2-x\right)}{dx} = 2x-1$

Proof:

$$\begin{aligned}\frac{d\left(x^2-x\right)}{dx} &= \lim_{h\to 0}\frac{\left((x+h)^2-(x+h)\right)-\left(x^2-x\right)}{h} = \lim_{h\to 0}\frac{x^2+2hx+h^2-x-h-x^2+x}{h}\\ &= \lim_{h\to 0}\frac{2hx+h^2-h}{h} = \lim_{h\to 0}\frac{h(2x+h-1)}{h} = \lim_{h\to 0} 2x+h-1 = 2x-1\end{aligned}$$

11. Claim: $f(x) = \sin^{-1} x$ then $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$

Solution: Recall that when we compose a function f with its inverse $f^{-1,}$ the result is always the same function:

$$f\left(f^{-1}(x)\right) = x$$

We will state this fact for $f(x) = \sin x$ and differentiate both sides of the equation. For the left-hand side, we use the chain rule.

$$\begin{aligned} \sin\left(\sin^{-1} x\right) &= x \\ \cos\left(\sin^{-1} x\right) \cdot \frac{d\left(\sin^{-1} x\right)}{dx} &= 1 \\ \frac{d\left(\sin^{-1} x\right)}{dx} &= \frac{1}{\cos\left(\sin^{-1} x\right)} = \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

To prove that $\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$, see problem 5b.

12. b) Recall that $\displaystyle\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$

$$\int \frac{x^2}{x^2+1} dx = \int \frac{x^2+1-1}{x^2+1} dx = \int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} dx = \int 1 - \frac{1}{x^2+1} dx = \int 1 dx - \int \frac{1}{x^2+1} dx = x - \tan^{-1} x + C$$

 Last revised: January 13, 2012