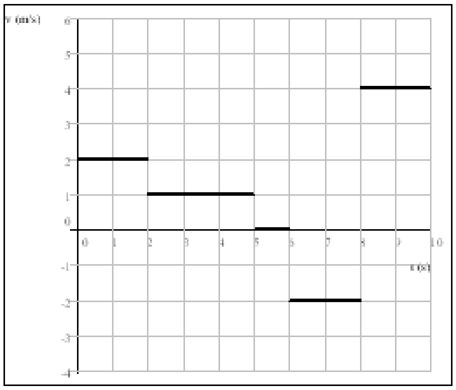
We have several reasons to want to know the area under the graph of a function. One reason is because this is how we obtaim area formulas. For example, the area of a circle is found by doubling the area under the graph of the function $f(x) = \sqrt{r^2 - x^2}$. Another example is when we need to multiply the dependent and independent variables (x and y) as the next example shows.



Example 1. Suppose that an object is moving along a vertical line. The graph shown depicts the velocity function $v(t)$ of the object. We know about the location function, $s(t)$ that $s(t) = 5$. What is the location of the object at $t = 10$?

Solution: This problem is quite easy, it is just strangely phrased. The graph is showing a velocity function that is constant on intervals. We also call a function like this a step-function.

During the first two seconds, the object has a constant velocity of $2\frac{\mathrm{m}}{\mathrm{s}}$: $v(t) = 2\frac{\mathrm{m}}{\mathrm{s}}$ on $[0\,\mathrm{s}, 2\,\mathrm{s}]$

On this interval, the distance traveled is $s = vt = 2\frac{\mathrm{m}}{\mathrm{s}} \cdot 2\,\mathrm{s} = 4\,\mathrm{m}$, thus the object traveled 4 meters upward between $t = 0\,\mathrm{s}$ and $t = 2\,\mathrm{s}$. If $s(0) = 5$, then $s(2) = 5 + 4 = 9$.

Next, between $t = 2\,\mathrm{s}$ and $t = 5\,\mathrm{s}$, the object has a constant velocity of $1\frac{\mathrm{m}}{\mathrm{s}}$: $v(t) = 1\frac{\mathrm{m}}{\mathrm{s}}$ on $[2\,\mathrm{s}, 5\,\mathrm{s}]$.

On this interval, the distance traveled is $s = vt = 1\frac{\mathrm{m}}{\mathrm{s}} \cdot 3\,\mathrm{s} = 3\,\mathrm{m}$, thus the object traveled 3 meters upward between $t = 2\,\mathrm{s}$ and $t = 5\,\mathrm{s}$. If $s(2) = 9$, then $s(5) = 9 + 3 = 12$.

During the next second, on $[5\,\mathrm{s}, 6\,\mathrm{s}]$, the object is at rest. Thus $s(6) = 12 + 0 = 12$.

Next, between $t = 6\,\mathrm{s}$ and $t = 8\,\mathrm{s}$, the object has a constant velocity of $-2\frac{\mathrm{m}}{\mathrm{s}}$. The negative velocity indicates that the object is moving downward. $v(t) = -2\frac{\mathrm{m}}{\mathrm{s}}$ on $[6\,\mathrm{s}, 8\,\mathrm{s}]$.

On this interval, the distance traveled is $s = vt = -2\frac{\mathrm{m}}{\mathrm{s}} \cdot 2\,\mathrm{s} = -4\,\mathrm{m}$, thus the object traveled 4 meters downward on $[6\,\mathrm{s}, 8\,\mathrm{s}]$. If $s(6) = 12$, then $s(8) = 12 - 4 = 8$.

Finally, between $t = 8\,\mathrm{s}$ and $t = 10\,\mathrm{s}$, object has a constant velocity of $4\frac{\mathrm{m}}{\mathrm{s}}$: $v(t) = 4\frac{\mathrm{m}}{\mathrm{s}}$ on $[8\,\mathrm{s}, 10\,\mathrm{s}]$.

On this interval, the distance traveled is $s = vt = 4\frac{\mathrm{m}}{\mathrm{s}} \cdot 2\,\mathrm{s} = 8\,\mathrm{m}$, thus the object traveled 8 meters upward on $[8\,\mathrm{s}, 10\,\mathrm{s}]$. If $s(8) = 8$, then $s(10) = 8 + 8 = 16$. Thus $\boxed{s(10) = 16}$.

This example shows that the distance traveled has a geometric interpretation, we are adding (signed) areas of rectangles. So we again have a reason for wanting to find the area under the graph.

In this sense, the area under the graph represents multiplication. Every time we need to multiply the dependent and independent variables (time and velocity in this case) and the dependent variable is not a constant, then the product becomes the area under the graph.

It is difficult to imagine an object whose velocity changes in no time from $-2\frac{\mathrm{m}}{\mathrm{s}}$ to $4\frac{\mathrm{m}}{\mathrm{s}}$. Our intuition suggests that the velocity graph of an object should be continuous. However, the following construction will be a method of finding the area under graphs of functions that are not necessarily continuous; so this concept of area will be more general than the area known in geometry.

Recall first the definition of a bounded function.

Definition: A function is **bounded above** if there exists a real number B such that for all x in domain, $f(x) \leq B$. We say that B is an upper bound for f.

Similarly, a function is **bounded below** if there exists a real number S such that for all x in domain, $f(x) \geq S$. We say that S is a lower bound for f.

A function is **bounded** if it is bounded from above and from below.

Suppose that f is a function bounded on a closed interval $[a,b]$. Boundedness is a nexessary but not sufficient condition on the existence of area under the graph. In other words, we will not even try to find area under the graph if f is not bounded on $[a,b]$.

Definition: A **partition** of $[a,b]$ is a finite set of numbers within the interval, listed in an increasing order. We usually denote a partition of $[a,b]$ by $\{a = x_0, x_1, x_2, ..., x_n = b\}$. Partitions divide a single interval into several smaller intevals. In the case of $\{a = x_0, x_1, x_2, ..., x_n = b\}$, the partition creates n sub-intervals. We often denote the length of the kth sub-interval by Δx_k.

Consider the five-unit long interval $[7,12]$. The partition $P = \{7,8,9,10,11,12\}$, defined by six numbers, partitions the interval $[7,12]$ into five smaller intervals: $[7,8]$, $[8,9]$, $[9,10]$,$[10,11]$, and $[11,12]$. Because we need six numers to define the five intervals, it is smart to start counting the numbers in the partition starting with zero. Assuming that each sub-interval has the same length, $\Delta x_k = \dfrac{5}{5} = 1$ for $k = 1,2,3,4,5$.

$$\{7,8,9,10,11,12\} \text{ where } 7 = x_0,\, 8 = x_1,\, 9 = x_2,\, 10 = x_3,\, 11 = x_4, \text{ and } 12 = x_5$$

The same interval can be divided (or partitioned) into ten smaller intervals. Assuming that each sub-interval has the same length, $\Delta x_k = \dfrac{5}{10} = \dfrac{1}{2}$ for $k = 1,2,3,...,10$.

$$\begin{aligned} \{x_0, x_1, x_2, ..., x_9, x_{10}\} &= \{7, 7.5, 8, ..., 11.5, 12\} \\ x_k &= 7 + 0.5k \text{ for } k = 1, 2.., 10 \end{aligned}$$

or into 100 smaller intervals: $\Delta x_k = \dfrac{5}{100} = 0.05$ for $k = 1,2,3,...,100$.

$$\begin{aligned} \{x_0, x_1, x_2, ..., x_{99}, x_{100}\} &= \{7, 7.05, 7.1, 7.15..., 11.5, 12\} \\ x_k &= 7 + 0.05k \text{ where } k = 0, 1, 2, ..., 100 \end{aligned}$$

The partitions we have seen divided the interval into smaller intervals of the same length. This is not necessary but often useful.

Definition: A partition that defines sub-intervals of the same length is called a **uniform partition**. In such cases, we use Δx to denote the common length of the sub-intervals Δx_k.

The length of subintervals in a uniform partition of $[a,b]$ with n subintervals is $\Delta x = \dfrac{b-a}{n}$.

Example 2. Suppose that $a = 2$ and $b = 4$. Find the uniform partition of $[a, b]$ of 50 subintervals.

Solution: The length of the entire interval is 2. Then the length of each sub-interval is $\dfrac{b-a}{n} = \dfrac{2}{50} = 0.04$. Then

$$a = x_0 = 2, x_1 = 2.04, x_2 = 2.08, \ldots x_{49} = 3.96, x_{50} = 4 = b \text{ or } x_k = 2 + k \cdot 0.04 \text{ for } k = 0, 1, 2, .., 50$$

This partition can also be expressed as $\{x_k = 2 + 0.04k \text{ for all } k = 0, 1, 2, ..50\}$

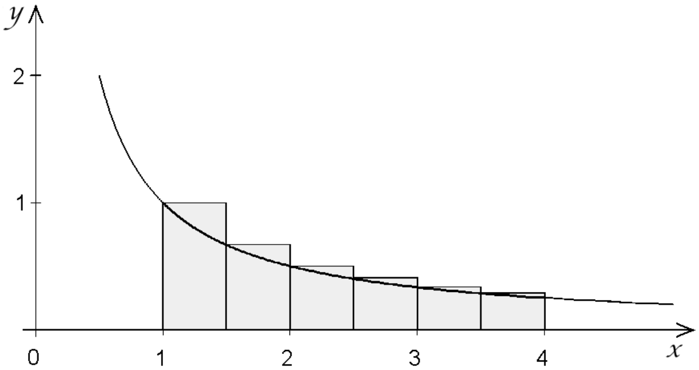
Suppose that f is a function bounded on $[a, b]$ and P is a partition of $[a, b]$ of n sub-intervals. We will approximate the area under the graph by approximating f as constant over each subinterval. This way the area under the graph is a rectangle. There are several reasonable ways of approximating f.

For example, let $f(x) = \dfrac{1}{x}$ on the interval $[1, 4]$. Let us use a uniform partition with $n = 6$.

On each of the interval $[x_{k,} x_{k+1}]$ of the partition, we will approximate f as constant. The value of that constant function will be the function value $f(x_k)$ of the left end-point in the interval.

On the first subinterval $[1, 1.5]$, we use $f(1)$ as the constant value. On the second subinterval $[1.5, 2]$, we use $f(1.5)$ as the constant value, and so on.

This way, we approximate the area by adding areas of rectangles. The horizontal side (we'll call it width) of each rectangle is $\dfrac{3}{6} = \dfrac{1}{2}$.



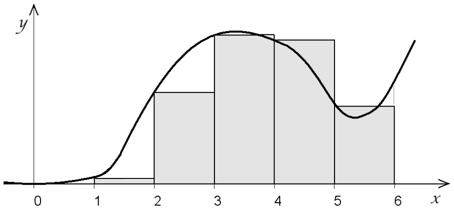
The heights of the six rectangles are $f(1)$, $f(1.5)$, $f(2)$, $f(2.5)$, $f(3)$, and $f(3.5)$. So the six areas added:

$$\begin{aligned}
A_{\text{appr}} &= \frac{1}{2} \cdot f(1) + \frac{1}{2} \cdot f(1.5) + \frac{1}{2} \cdot f(2) + \frac{1}{2} \cdot f(2.5) + \frac{1}{2} \cdot f(3) + \frac{1}{2} \cdot f(3.5) \\
&= \frac{1}{2} \left(f(1) + f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) \right) \\
&= \frac{1}{2} \left(f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right) \right) \qquad f(x) = \frac{1}{x} \\
&= \frac{1}{2} \left(1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7} \right) \approx 1.592\,86
\end{aligned}$$

Such an approximation is called a **left Riemann sum**.

We did not find the exact area. Instead, we have a somewhat crude approximation of it. In this case, our approximation is certainly above the exact value, because the function is completely covered by our rectangles on $[1, 4]$. In short, our approximation *over-estimates* the area.

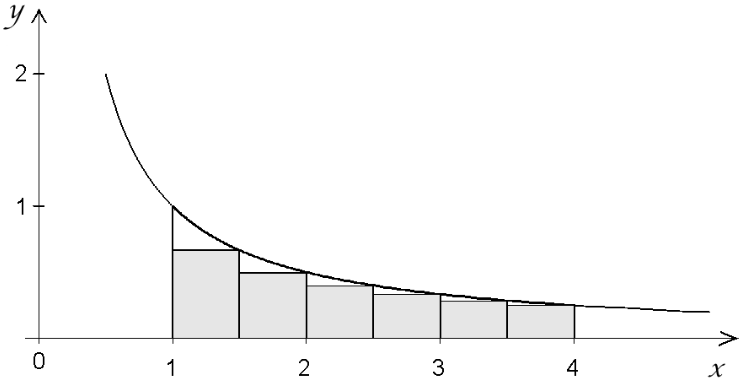
This is not always the case. It happens in case of $f(x) = \dfrac{1}{x}$ because it is a decreasing function. Using a left Riemann sum for other functions may overestimate or underestimate the area under the graph, and in some cases we cannot know which.



Similarly, we can approximate the function over each subinterval as the right end-point.

In this case, the height of the first rectangle will be $f(1.5)$, that of the second rectangle $f(2)$, and so on. The approximate area is then

$$A_{\text{appr}} = \sum_{k=1}^{6} f(x_k)\,\Delta x$$



$$\begin{aligned}
A_{\text{appr}} &= \frac{1}{2}\cdot f(1.5) + \frac{v}{2}\cdot f(2) + \frac{1}{2}\cdot f(2.5) + \frac{1}{2}\cdot f(3) + \frac{1}{2}\cdot f(3.5) + \frac{1}{2}\cdot f(4) \\
&= \frac{1}{2}\left(f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4)\right) = \frac{1}{2}\left(f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right) + f(4)\right) \\
&= \frac{1}{2}\left(\frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7} + \frac{1}{4}\right) \approx 1.217\,86
\end{aligned}$$

This approximation uses the left end point in each sub-interval to estimate the function value. A sum like this is called a **right Riemann sum**.

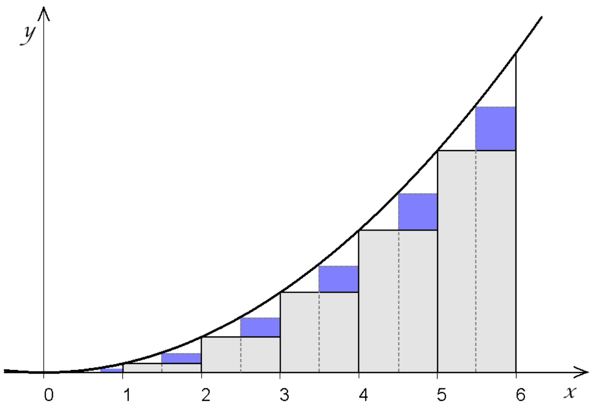
There are different Riemann sums. The midpoint Riemann sum uses the function value taken at the midpoint of each subinterval. Another Riemann sum would approximate the function as the average of the function values taken at the two end points of the subinterval.

We have only assumed that f is bounded. But if f is also continuous, then the function takes an absolute maximum and minimum in each subinterval. A Riemann sum that uses the minimum in each subinterval obviously underestimates the area under the graph. A Riemann sum that uses the maximum in each subinterval obviously overestimates the area under the graph. We will later see the usefulness of Riemann sums in cases we do know whether the estmation is greater than the area or smaller.

But how do we get better results than crude approximations? One way to improve the approximate values is to refine the partition.

Definition: A partition $P = \{x_0, x_1, x_2, ..., x_n\}$ is **refined** if we keep all numbers from P and add more.

As we refine partitions, the estimation for the area can only improve. Under-estimations increase, over-estimations decrease. The picture shows a left Riemann sum on a uniform partition and the same Riemann sum on a finer partition. The blue regions are the improvement in the approximation of the area.



Suppose that P_1, P_2,...... is a sequence of partition, each a refinement of the previous one. Let us also suppose that as we refine the partitions, we divide each sub-interval, i.e. the length of the longest sub interval approaches zero. For each partition P_k, let U_k be an underestimation of the area under the graph, and O_k an overestimation of the same area with the same partition P_k.

Consider now the sequences O_1, O_2,... and U_1, U_2, ... Every over-estimation of the area is naturally an upper bound for any under-estimation of the area under the graph.

$$O_n \geq U_m \quad \text{for all } n, m \text{ natural numbers.}$$

Since refining partitions can only improve estimations, the sequence O_1, O_2,... is decreasing and the sequence U_1, U_2, ... is increasing.

The sequence O_1, O_2,... is decreasing and is bounded below by any U_k. Therefore, it must have a limit, L. The sequence U_1, U_2,... is increasing and is bounded above by any O_k. Therefore, it must have a limit, M, where $L \geq M$. If $L > M$, we say that the area under the graph doesn't exist. If $L = M$, we define this common limit to be the area under the graph. If this area exists, we say that f is **integrable** on $[a, b]$.

The following theorems will not be proven, but we should state them here.

Theorem: If the area under the graph exists, it is the same, no matter what kind of partitions and Riemann sums we use.

Theorem: If a function f is continuous on $[a, b]$, then it is integrable there.

Theorem: If a function f has only finitely many discontinuities on $[a, b]$, then it is integrable there.

Theorem: If a function f is increasing on $[a, b]$, then it is integrable there.

Sample Problems

1. Consider the function $f(x) = \dfrac{1}{x}$ on the interval $[1, 4]$.
   a) Compute the left Riemann sum for f on this interval using a regular partition with $n = 6$ subintervals.
   b) Compute the right Riemann sum for f on this interval using a regular partition with $n = 6$ subintervals.
2. Consider the function $f(x) = x^2$ on the interval $[0, 6]$.
   a) Compute the left Riemann sum for f on this interval with $n = 6$ subintervals.
   b) Compute the right Riemann sum for f on this interval with $n = 6$ subintervals.
   c) Compute the left Riemann sum for f on this interval with $n = 12$ subintervals.
   d) Compute the right Riemann sum for f on this interval with $n = 12$ subintervals.
   e) Compute the left Riemann sum for f on this interval with $n = 100$ subintervals.
   f) Compute the right Riemann sum for f on this interval with $n = 100$ subintervals.
   g) Compute the left Riemann sum for f on this interval with n subintervals.
   h) Compute the limit of the left Riemann sum for f on this interval with n intervals, as n approaches infinity.
   i) Compute the right Riemann sum for f on this interval with n subintervals.
   j) Compute the limit of the right Riemann sum for f on this interval with n intervals, as n approaches infinity.

Practice Problems

1. Consider the function $f(x) = \sqrt{x}$ on the interval $[0,4]$.
    a) Compute the left Riemann sum for f on this interval with $n = 4$ subintervals.
    b) Compute the right Riemann sum for f on this interval with $n = 4$ subintervals.
    c) Compute the left Riemann sum for f on this interval with $n = 10$ subintervals.
    d) Compute the right Riemann sum for f on this interval with $n = 10$ subintervals.

2. Consider the function $f(x) = \ln(x+1)$ on the interval $[0,10]$.
    a) Compute the left Riemann sum for f on this interval with $n = 10$ subintervals.
    b) Compute the right Riemann sum for f on this interval with $n = 10$ subintervals.

3. Consider the function $f(x) = x^3$ on the interval $[0,1]$.
    a) Compute the left Riemann sum for f on this interval with $n = 4$ subintervals.
    b) Compute the right Riemann sum for f on this interval with $n = 4$ subintervals.
    c) Compute the left Riemann sum for f on this interval with $n = 10$ subintervals.
    d) Compute the right Riemann sum for f on this interval with $n = 10$ subintervals.
    e) Compute the left Riemann sum for f on this interval with $n = 100$ subintervals.
    f) Compute the right Riemann sum for f on this interval with $n = 100$ subintervals.

Answers

Sample Problems

1. a) $\frac{223}{140} \approx 1.592\,857$ b) $\frac{341}{280} \approx 1.217\,857$

2. a) 55 b) 91 c) $\frac{253}{4} = 63.25$ d) $\frac{325}{4} = 81.25$ e) $\frac{177\,309}{2500} = 70.924$

   f) $\frac{182\,709}{2500} = 73.084$ g) $\frac{36\,(n-1)\,(2n-1)}{n^2} = \frac{72n^2 - 108n + 36}{n^2}$ h) 72

   i) $\frac{36\left(2n^2 + 3n + 1\right)}{n^2} = \frac{72n^2 + 108n + 36}{n^2}$ j) 72

Practice Problems

1. a) $1 + \sqrt{2} + \sqrt{3} \approx 4.146\,264\,37$ b) $3 + \sqrt{2} + \sqrt{3} \approx 6.146\,264\,37$

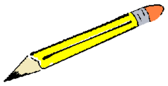
   c) $\frac{2\sqrt{10}}{25}\left(6 + \sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8}\right) \approx 4.884\,075$

   d) $\frac{2\sqrt{10}}{25}\left(6 + \sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8} + \sqrt{10}\right) \approx 5.684\,075$

2. a) $\ln(10!) = \ln 3628\,800 \approx 15.104\,412\,57$ b) $\ln(11!) = \ln 39\,916\,800 \approx 17.502\,307\,85$

3. a) $\frac{9}{64} = 0.140\,625$ b) $\frac{25}{64} = 0.390\,625$ c) $\frac{81}{400} = 0.202\,5$ d) $\frac{121}{400} = 0.302\,5$

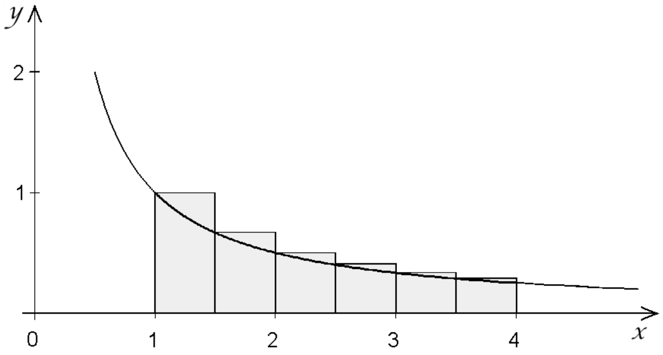
4. e) $\frac{9801}{40\,000} = 0.245\,025$ f) $\frac{10\,201}{40\,000} = 0.255\,025$

Sample Problems  Solutions

1. Consider the function $f(x) = \dfrac{1}{x}$ on the interval $[1,4]$.

   a) Compute the left Riemann sum for f on this interval using a regular partition with $n = 6$ subintervals.

   Solution: The interval $[1,4]$ is 3 units long. The regular partition will contain intervals of length $\dfrac{3}{6} = \dfrac{1}{2}$. The partition consists of $\left\{1,\ 1\frac{1}{2},2,\ 2\frac{1}{2},3,\ 3\frac{1}{2},\ 4\right\}$ Notice that these are seven numbers. We usually start labeling with zero. In this case, these seven numbers are $\{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$. On each interval, we approximate the area under the graph by a rectangle as tall as the function value of the left endpoint of the interval. For example, on the first interval, we approximate the area under the graph using a rectangle with height $\dfrac{1}{1} = 1$. On the second interval, the height of the rectangle is $\dfrac{1}{1\frac{1}{2}} = \dfrac{2}{3}$.



   The first rectangle has width $\dfrac{1}{2}$ and height 1. The area is $A_1 = \dfrac{1}{2} \cdot 1 = \dfrac{1}{2}$.

   The second rectangle has width $\dfrac{1}{2}$ and height $\dfrac{1}{\left(\frac{3}{2}\right)} = \dfrac{2}{3}$. The area is $A_2 = \dfrac{1}{2} \cdot \dfrac{2}{3} = \dfrac{1}{3}$.

   Let us notice that all rectangles have the same width of $\dfrac{1}{2}$. This is an advantage of a regular partition.

   The third rectangle has height $\dfrac{1}{2}$. Its area is $A_3 = \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{4}$.

   The fourth rectangle has height $\dfrac{1}{\left(\frac{5}{2}\right)} = \dfrac{2}{5}$. Its area is $A_4 = \dfrac{1}{2} \cdot \dfrac{2}{5} = \dfrac{1}{5}$

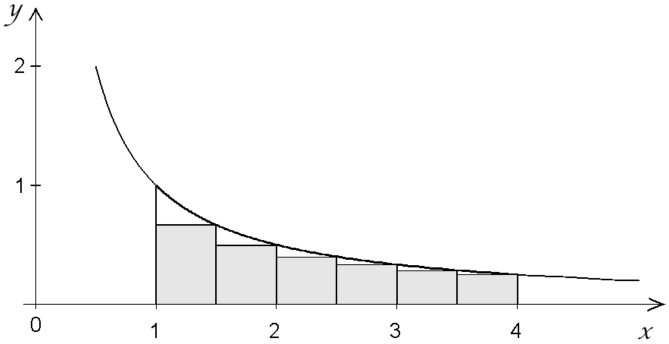
   The fifth rectangle has height $\dfrac{1}{3}$. Its area is $A_4 = \dfrac{1}{2} \cdot \dfrac{1}{3} = \dfrac{1}{6}$.

   The sixth rectangle has height $\dfrac{1}{\left(\frac{7}{2}\right)} = \dfrac{2}{7}$. Its area is $A_4 = \dfrac{1}{2} \cdot \dfrac{2}{7} = \dfrac{1}{7}$. In short, the left-hand approximation is

$$
\begin{aligned}
L_{f,n=6} &= \frac{1}{2}\cdot\frac{1}{1} + \frac{1}{2}\cdot\frac{1}{1.5} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2.5} + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3.5} = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{1.5} + \frac{1}{2} + \frac{1}{2.5} + \frac{1}{3} + \frac{1}{3.5}\right) \\
&= \frac{1}{2}\left(1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7}\right) = \frac{223}{140} \approx 1.592\,857
\end{aligned}
$$

   We can clearly see from the picture that this approximation is an overestimation of the area.

b) Compute the right Riemann sum for f on this interval using a regular partition with $n = 6$ subintervals.



$$R = \frac{1}{2} \cdot \frac{1}{1.5} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2.5} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3.5} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2}\left(\frac{1}{1.5} + \frac{1}{2} + \frac{1}{2.5} + \frac{1}{3} + \frac{1}{3.5} + \frac{1}{4}\right)$$
$$= \frac{1}{2}\left(\frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7} + \frac{1}{4}\right) = \frac{341}{280} \approx 1.217\,857$$

We can clearly see from the picture that this approximation is an underestimation of the area. Thus we now know that the area under the graph is between those two values:

$$1.217\,857 < A < 1.592\,857$$

Note: We sometimes use summation notation when writing such expressions. Using summation notation, these Riemann sums are

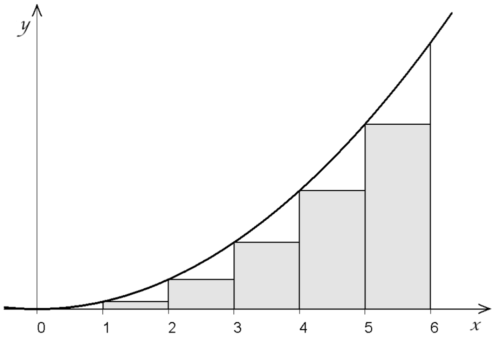
$$L = \sum_{k=0}^{5} \frac{1}{2} \cdot \frac{1}{1 + k\left(\frac{1}{2}\right)} = \frac{1}{2} \sum_{k=0}^{5} \frac{1}{1 + k\left(\frac{1}{2}\right)} = \frac{1}{2} \sum_{k=0}^{5} \frac{1}{\frac{2+k}{2}} = \frac{1}{2} \sum_{k=0}^{5} \frac{2}{2+k} = \sum_{k=0}^{5} \frac{1}{2+k}$$

$$\text{and } R = \sum_{k=1}^{6} \frac{1}{2} \cdot \frac{1}{1 + k\left(\frac{1}{2}\right)} = \sum_{k=1}^{6} \frac{1}{2+k}$$

2. For this problem, we will need the following theorem: for all natural numbers n,

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Consider the function $f(x) = x^2$ on the interval $[0, 6]$.
a) Compute the left Riemann sum for f on this interval with $n = 6$ subintervals.



Solution: Each subinterval is of length 1, and so the partition is $\{0, 1, 2, 3, 4, 5, 6\}$. The left-hand sum is

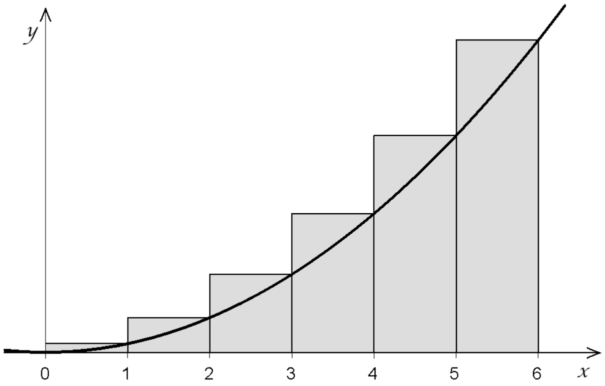
$$L_{f,n=6} = 1 \cdot 0^2 + 1 \cdot 1^2 + 1 \cdot 2^2 + 1 \cdot 3^2 + 1 \cdot 4^2 + 1 \cdot 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

Using summation notation,

$$L_{f,n=6} = \sum_{k=0}^{5} 1 \cdot k^2 = \sum_{k=0}^{5} k^2 = 55$$

We can see on the picture that this Riemann sum underestimates the area.

b) Compute the right Riemann sum for f on this interval with $n = 6$ subintervals.



$$R_{f,n=6} = 1\cdot 1^2 + 1\cdot 2^2 + 1\cdot 3^2 + 1\cdot 4^2 + 1\cdot 5^2 + 1\cdot 6^2 = 1 + 4 + 9 + 16 + 25 + 36 = 91$$

Using summation notation,

$$R_{f,n=6} = \sum_{k=1}^{6} 1\cdot k^2 = \sum_{k=1}^{6} k^2 = 91$$

We can see on the picture that this Riemann sum underestimates the area. Thus, we have that

$$55 < A < 91$$

c) Compute the left Riemann sum for f on this interval with $n = 12$ subintervals.

Solution: Each subinterval will have length $\dfrac{6}{12} = \dfrac{1}{2}$. The partition is $\{0,\ 0.5,\ 1,\ 1.5\ ,2\ ,2.5,\ 3,\ 3.5,\ 4,\ 4.5,\ 5,\ 5.5,\ 6\}$. The left Riemann sum is

$$\begin{aligned} L_{f,n=12} \quad = \quad & \frac{1}{2}\cdot 0^2 + \frac{1}{2}\cdot 0.5^2 + \frac{1}{2}\cdot 1^2 + \frac{1}{2}\cdot 1.5^2 + \frac{1}{2}\cdot 2^2 + \frac{1}{2}\cdot 2.5^2 + \frac{1}{2}\cdot 3^2 + \frac{1}{2}\cdot 3.5^2 + \frac{1}{2}\cdot 4^2 \\ & + \frac{1}{2}\cdot 4.5^2 + \frac{1}{2}\cdot 5^2 + \frac{1}{2}\cdot 5.5^2 \end{aligned}$$

Although this looks like a lot of computation, it can be made quite simple using a bit of algebra and the theorem stated above. We first factor out $\dfrac{1}{2}$ and write the rest as fractions, with a common denominator of 2.

$$\begin{aligned} L_{f,n=12} &= \frac{1}{2}\left(0^2 + 0.5^2 + 1^2 + 1.5^2 + \ldots + 5.5^2\right) = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{4}{2}\right)^2 + \ldots + \left(\frac{11}{2}\right)^2\right) \\ &= \frac{1}{2}\left(\frac{1^2}{4} + \frac{2^2}{4} + \frac{3^2}{4} + \frac{4^2}{4} + \ldots + \frac{11^2}{4}\right) \qquad \text{factor out } \frac{1}{4} \\ &= \frac{1}{2}\cdot\frac{1}{4}\left(1^2 + 2^2 + 3^2 + \ldots + 11^2\right) \qquad \text{use theorem with } n = 11 \\ &= \frac{1}{8}\cdot\frac{11\cdot 12\cdot 23}{6} = \frac{253}{4} = 63.25 \end{aligned}$$

The same computation, using summation notation:

$$L_{f,n=12} = \sum_{k=0}^{11} \frac{1}{2}\cdot\left(\frac{1}{2}k\right)^2 = \frac{1}{2}\sum_{k=0}^{11}\frac{k^2}{4} = \frac{1}{8}\sum_{k=0}^{11} k^2 = \frac{1}{8}\frac{11\cdot 12\cdot 23}{6} = \frac{253}{4} = 63.25$$

d) Compute the right Riemann sum for f on this interval with $n = 12$ subintervals.

Solution: The difference between the left and right Riemann sums is just the first and the last rectangle.

$$\begin{aligned}
R_{f,n=12} &= \frac{1}{2}\left(0.5^2 + 1^2 + \dots + 5.5^2 + 6^2\right) = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{4}{2}\right)^2 + \dots + \left(\frac{12}{2}\right)^2\right) \\
&= \frac{1}{2}\left(\frac{1^2}{4} + \frac{2^2}{4} + \frac{3^2}{4} + \frac{4^2}{4} + \dots + \frac{12^2}{4}\right) \qquad \text{factor out } \frac{1}{4} \\
&= \frac{1}{2} \cdot \frac{1}{4}\left(1^2 + 2^2 + 3^2 + \dots + 12^2\right) \qquad \text{use theorem with } n = 12 \\
&= \frac{1}{8} \cdot \frac{12 \cdot 13 \cdot 25}{6} = \frac{325}{4} = 81.25
\end{aligned}$$

Using summation notation,

$$R_{f,n=12} = \sum_{k=1}^{12} \frac{1}{2} \cdot \left(\frac{1}{2}k\right)^2 = \frac{1}{2}\sum_{k=1}^{12} \frac{k^2}{4} = \frac{1}{8}\sum_{k=1}^{12} k^2 = \frac{1}{8}\frac{12 \cdot 13 \cdot 25}{6} = \frac{325}{4} = 81.25$$

Because this function is increasing, all left sums underestimate the area and all right sums overestimate the area under the graph. Thus

$$63.25 < A < 81.25$$

e) Compute the left Riemann sum for f on this interval with $n = 100$ subintervals.

Solution: Each subinterval is $\frac{6}{100}$ units long. The partition is

$$\left\{x_0 = 0,\ x_1 = \frac{6}{100},\ x_2 = \frac{12}{100},\ \dots, x_k = \frac{6k}{100}, \dots, x_{100} = \frac{600}{100} = 6\right\}$$

The left Riemann sum is

$$\begin{aligned}
L_{f,n=100} &= \frac{6}{100} \cdot 0^2 + \frac{6}{100} \cdot \left(\frac{6}{100}\right)^2 + \frac{6}{100} \cdot \left(\frac{12}{100}\right)^2 + \dots + \frac{6}{100} \cdot \left(\frac{6 \cdot 99}{100}\right)^2 \\
&= \frac{6}{100}\left(0^2 + \left(\frac{6 \cdot 1}{100}\right)^2 + \left(\frac{6 \cdot 2}{100}\right)^2 + \dots + \left(\frac{6 \cdot 99}{100}\right)^2\right) \\
&= \frac{6}{100} \cdot \left(\frac{6}{100}\right)^2 \left(0^2 + 1^2 + 2^2 + \dots + 99^2\right) = \left(\frac{6}{100}\right)^3 \frac{99 \cdot 100 \cdot 199}{6} \\
&= \left(\frac{6}{100}\right)^2 (99 \cdot 199) = \frac{177\,309}{2500} = 70.924
\end{aligned}$$

Using summation notation,

$$\begin{aligned}
L_{f,n=100} &= \sum_{k=0}^{99} \frac{6}{100} \cdot \left(\frac{6}{100}k\right)^2 = \frac{6}{100}\sum_{k=0}^{99}\left(\frac{6}{100}\right)^2 k^2 = \left(\frac{6}{100}\right)^3 \sum_{k=0}^{99} k^2 = \left(\frac{6}{100}\right)^3 \frac{99 \cdot 100 \cdot 199}{6} \\
&= \left(\frac{6}{100}\right)^2 (99 \cdot 199) = \frac{177\,309}{2500} = 70.924
\end{aligned}$$

f) Compute the right Riemann sum for f on this interval with $n = 100$ subintervals.

The right Riemann sum is

$$\begin{aligned}
R_{f,n=100} &= \frac{6}{100}\cdot\left(\frac{6}{100}\right)^2 + \frac{6}{100}\cdot\left(\frac{12}{100}\right)^2 + \text{....} + \frac{6}{100}\cdot\left(\frac{6\cdot 100}{100}\right)^2 \\
&= \frac{6}{100}\left(\left(\frac{6\cdot 1}{100}\right)^2 + \left(\frac{6\cdot 2}{100}\right)^2 + \text{....} + \left(\frac{6\cdot 100}{100}\right)^2\right) \\
&= \frac{6}{100}\cdot\left(\frac{6}{100}\right)^2\left(0^2+1^2+2^2+\text{....}+100^2\right) = \left(\frac{6}{100}\right)^3\frac{100\cdot 101\cdot 201}{6} \\
&= \left(\frac{6}{100}\right)^2(101\cdot 201) = \frac{182\,709}{2500} = 73.083\,6
\end{aligned}$$

Using summation notation,

$$\begin{aligned}
R_{f,n=100} &= \sum_{k=1}^{100}\frac{6}{100}\cdot\left(\frac{6}{100}k\right)^2 = \frac{6}{100}\sum_{k=1}^{100}\left(\frac{6}{100}\right)^2 k^2 = \left(\frac{6}{100}\right)^3\sum_{k=1}^{100}k^2 = \left(\frac{6}{100}\right)^3\frac{100\cdot 101\cdot 201}{6} \\
&= \left(\frac{6}{100}\right)^2(100\cdot 201) = \frac{182\,709}{2500} = 73.084
\end{aligned}$$

Thus

$$70.924 < A < 73.084$$

g) Compute the left Riemann sum for f on this interval with n subintervals.

Solution: Each subinterval is $\frac{6}{n}$ units long. The numbers in the partition are

$$\left\{x_0 = 0,\ x_1 = \frac{6}{n},\ x_2 = 2\left(\frac{6}{n}\right),\ x_3 = 3\left(\frac{6}{n}\right),\ x_4 = 4\left(\frac{6}{n}\right), \text{....},\ x_n = n\left(\frac{6}{n}\right) = 6\right\}$$

The left Riemann sum is

$$\begin{aligned}
L_{f,n} &= \frac{6}{n}\cdot 0^2 + \frac{6}{n}\left(\frac{6}{n}\right)^2 + \frac{6}{n}\left(2\cdot\frac{6}{n}\right)^2 + \frac{6}{n}\left(3\cdot\frac{6}{n}\right)^2 + \text{....} + \frac{6}{n}\left((n-1)\cdot\frac{6}{n}\right)^2 \\
&= \frac{6}{n}\left(0^2 + \left(\frac{6}{n}\right)^2 + \left(2\cdot\frac{6}{n}\right)^2 + \left(3\cdot\frac{6}{n}\right)^2 + \text{....} + \left((n-1)\cdot\frac{6}{n}\right)^2\right) \\
&= \frac{6}{n}\left(1^2\cdot\left(\frac{6}{n}\right)^2 + 2^2\cdot\left(\frac{6}{n}\right)^2 + 3^2\cdot\left(\frac{6}{n}\right)^2 + \text{....} + (n-1)^2\cdot\left(\frac{6}{n}\right)^2\right) \\
&= \frac{6}{n}\left(\frac{6}{n}\right)^2\left(1^2+2^2+3^2+\text{....}+(n-1)^2\right) = \left(\frac{6}{n}\right)^3\frac{(n-1)\left((n-1)+1\right)\left(2(n-1)+1\right)}{6} \\
&= \left(\frac{6}{n}\right)^3\frac{(n-1)\,n\,(2n-1)}{6} = \left(\frac{6}{n}\right)^2(n-1)(2n-1) = \frac{36(n-1)(2n-1)}{n^2} \\
&= \frac{36\left(2n^2-3n+1\right)}{n^2} = \frac{72n^2-108n+36}{n^2}
\end{aligned}$$

Using summation notation,

$$\begin{aligned}
L_{f,n} &= \sum_{k=0}^{n-1}\frac{6}{n}\cdot\left(\frac{6}{n}k\right)^2 = \frac{6}{n}\sum_{k=0}^{n-1}\left(\frac{6}{n}\right)^2 k^2 = \left(\frac{6}{n}\right)^3\sum_{k=0}^{n-1}k^2 = \left(\frac{6}{n}\right)^3\frac{(n-1)\left((n-1)+1\right)\left(2(n-1)+1\right)}{6} \\
&= \left(\frac{6}{n}\right)^3\frac{(n-1)\,n\,(2n-1)}{6} = \left(\frac{6}{n}\right)^2(n-1)(2n-1) = \frac{36(n-1)(2n-1)}{n^2} = \frac{72n^2-108n+36}{n^2}
\end{aligned}$$

h) Compute the limit of the left Riemann sum for f on this interval with n intervals, as n approaches infinity.

Solution:
$$\lim_{n\to\infty}\frac{72n^2-108n+36}{n^2}=\lim_{n\to\infty}\left(72-\frac{108}{n}+\frac{36}{n^2}\right)=72$$

i) Compute the right Riemann sum for f on this interval with n subintervals.

Solution: The right Riemann sum is

$$\begin{aligned}R_{f,n} &= \frac{6}{n}\left(\frac{6}{n}\right)^2+\frac{6}{n}\left(2\cdot\frac{6}{n}\right)^2+\frac{6}{n}\left(3\cdot\frac{6}{n}\right)^2+....+\frac{6}{n}\left(n\cdot\frac{6}{n}\right)^2\\ &= \frac{6}{n}\left(\left(\frac{6}{n}\right)^2+\left(2\cdot\frac{6}{n}\right)^2+\left(3\cdot\frac{6}{n}\right)^2+....+\left(n\cdot\frac{6}{n}\right)^2\right)\\ &= \frac{6}{n}\left(1^2\cdot\left(\frac{6}{n}\right)^2+2^2\cdot\left(\frac{6}{n}\right)^2+3^2\cdot\left(\frac{6}{n}\right)^2+....+n^2\cdot\left(\frac{6}{n}\right)^2\right)\end{aligned}$$

$$\begin{aligned}R_{f,n} &= \frac{6}{n}\left(\frac{6}{n}\right)^2\left(1^2+2^2+3^2+....+n^2\right)=\left(\frac{6}{n}\right)^3\frac{n\left(n+1\right)\left(2n+1\right)}{6}\\ &= \left(\frac{6}{n}\right)^2\left(n+1\right)\left(2n+1\right)=\frac{36\left(n+1\right)\left(2n+1\right)}{n^2}=\frac{36\left(2n^2+3n+1\right)}{n^2}=\frac{72n^2+108n+36}{n^2}\end{aligned}$$

Using summation notation,

$$\begin{aligned}R_{f,n} &= \sum_{k=1}^{n}\frac{6}{n}\cdot\left(\frac{6}{n}k\right)^2=\frac{6}{n}\sum_{k=1}^{n}\left(\frac{6}{n}\right)^2k^2=\left(\frac{6}{n}\right)^3\sum_{k=1}^{n}k^2=\left(\frac{6}{n}\right)^3\frac{n\left(n+1\right)\left(2n+1\right)}{6}\\ &= \left(\frac{6}{n}\right)^2\left(n+1\right)\left(2n+1\right)=\frac{36\left(n+1\right)\left(2n+1\right)}{n^2}=\frac{72n^2+108n+36}{n^2}\end{aligned}$$

j) Compute the limit of the right Riemann sum for f on this interval with n intervals, as n approaches infinity.

Solution:
$$\lim_{n\to\infty}\frac{72n^2+108n+36}{n^2}=\lim_{n\to\infty}\left(72+\frac{108}{n}+\frac{36}{n^2}\right)=72$$