

Definition: Given a sequence of numbers  $\{a_n\}$ , the expression

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

is called an **infinite series**. The number  $a_n$  is the  **$n$ th term** of the series. The sequence  $\{s_n\}$  defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^n a_k \end{aligned}$$

is called the **sequence of partial sums** of the series, the number  $s_n$  being the  **$n$ th partial sum**. If the sequence of partial sums converges to a limit  $L$ , we say that the series converges and that its **sum** is  $L$ . In this case, we also write

$$a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Example 1)  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$  is convergent.

Proof: The series is  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$ . Consider the sequence  $\{s_n\}$  of partial sums.

$$\begin{aligned} s_1 &= \frac{1}{2} & s_4 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5} \\ s_2 &= \frac{1}{2} + \frac{1}{6} = \frac{2}{3} & s_5 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6} \\ s_3 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4} & s_6 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} = \frac{6}{7} \end{aligned}$$

It appears that  $s_n = \frac{n}{n+1}$ . If that was so, then the sum of the series can be easily found as the limit of the sequence of partial sums:

$$s = \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

This is the case indeed. We can prove it in general using partial fractions:

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

and so

$$s_1 = a_1 = \frac{1}{1} - \frac{1}{2} = 1 - \frac{1}{2}$$

$$s_2 = a_1 + a_2 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$s_3 = a_1 + a_2 + a_3 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$s_n = a_1 + a_2 + \dots + a_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

In this case, each partial sum of the series has only a fixed number of terms after cancellation. We call such a series a **telescoping sum**.

Definition: **Geometric series** are of the form

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$

where  $a$  and  $r$  are fixed real numbers and  $a \neq 0$ .

Theorem: If  $|r| < 1$ , the geometric series  $a + ar + ar^2 + \dots + ar^{n-1} + \dots$  converges to  $\frac{a}{1-r}$  and if  $|r| \geq 1$ , the series diverges.

Example 2) Determine whether the given series converges or diverges. If it converges, find the sum of the series.

a)  $\sum_{n=0}^{\infty} 2^{n-1} \cdot 3^{2-n}$

We start by re-writing  $a_n$

$$a_n = 2^{n-1} \cdot 3^{2-n} = \frac{2^n}{2} \cdot \frac{9}{3^n} = \frac{9}{2} \left(\frac{2}{3}\right)^n$$

We can now determine that  $a = \frac{9}{2}$  and  $r = \frac{2}{3}$ . Since  $-1 < r < 1$ , the series converges and its sum is

$$s = \frac{a}{1-r} = \frac{\frac{9}{2}}{1 - \frac{2}{3}} = \frac{27}{2}$$

b)  $\sum_{n=3}^{\infty} 3^{2n-1} \cdot (-5)^{2-n}$

We start by re-writing  $a_n$

$$a_n = 3^{2n-1} \cdot (-5)^{2-n} = \frac{9^n}{3} \cdot \frac{25}{(-5)^n} = \frac{25}{3} \left(-\frac{9}{5}\right)^n$$

We can now determine that  $a = \frac{25}{3}$  and  $r = -\frac{9}{5}$ . Since  $r < -1$ , the series diverges and so the sum is not defined. There are more examples in the separate handout dedicated to geometric series.

Theorem: If  $\sum_{n=1}^{\infty} a_n$  converges, then  $a_n \rightarrow 0$ .

Proof: Let  $\sum_{n=1}^{\infty} a_n$  be a convergent series with sum  $s \in \mathbb{R}$ . Then  $\lim_{n \rightarrow \infty} s_n = s$ . Clearly  $a_n = s_n - s_{n-1}$ .

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = s - s = 0$$

Theorem: (*n*th term test) If  $\lim_{n \rightarrow \infty} a_n$  doesn't exist or exists but is not zero, then  $\sum_{n=1}^{\infty} a_n$  diverges.

The "smallness" of the *n*th term is a necessary but not sufficient condition of the convergence of the series. For example,  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges even though the *n*th term,  $a_n$  approaches zero. The logical conclusion is that the *n*th term test can only be used to prove divergence of a series.

Example 3) Prove that  $\sum_{n=1}^{\infty} \frac{n-2}{2n+1}$  diverges.

Proof:  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n-2}{2n+1} = \lim_{n \rightarrow \infty} \frac{1 - \frac{2}{n}}{2 + \frac{1}{n}} = \frac{1}{2}$ . Since the *n*th term does not approach zero, the series diverges.

Theorem: If  $\sum a_n = A$  and  $\sum b_n = B$  are convergent series, then

- 1) Sum Rule  $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
- 2) Difference Rule  $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
- 3) Constant Multiple Rule:  $\sum k a_n = k \sum a_n = kA$  for any  $k \in \mathbb{R}$

These properties can be proved using the properties of limits of sequences. Consequences:

- 1) Every non-zero constant multiple of a divergent series diverges.
- 2) If  $\sum a_n$  converges and  $\sum b_n$  diverges, then  $\sum (a_n + b_n)$  and  $\sum (a_n - b_n)$  both diverge.

Example 4.) Compute the sum of each of the following series or state if the series diverges.

a)  $\sum_{n=0}^{\infty} \frac{7 \cdot 2^n - 3^n}{5^n}$

Solution:

$$\sum_{n=0}^{\infty} \frac{7 \cdot 2^n - 3^n}{5^n} = \sum_{n=0}^{\infty} \left( \frac{7 \cdot 2^n}{5^n} - \frac{3^n}{5^n} \right) = \sum_{n=0}^{\infty} 7 \left( \frac{2^n}{5^n} \right) - \sum_{n=0}^{\infty} \left( \frac{3^n}{5^n} \right) = \sum_{n=0}^{\infty} 7 \left( \frac{2}{5} \right)^n - \sum_{n=0}^{\infty} \left( \frac{3}{5} \right)^n$$

We separately compute the sums of the geometric series. In the first series,  $a = 7$  and  $r = \frac{2}{5}$  and in the second,  $a = 1$  and  $r = \frac{3}{5}$ .

$$s_1 = \frac{7}{1 - \frac{2}{5}} = \frac{35}{3} \quad \text{and} \quad s_2 = \frac{1}{1 - \frac{3}{5}} = \frac{5}{2}$$

and so the difference of the two series is  $\frac{35}{3} - \frac{5}{2} = \frac{55}{6}$ .

b)  $\sum_{n=0}^{\infty} e^n + e^{-n}$

Solution:

$$\sum_{n=0}^{\infty} e^n + e^{-n} = \sum_{n=0}^{\infty} e^n + \sum_{n=0}^{\infty} e^{-n}$$

The first series diverges because it is a geometric series with  $r = e > 1$ . Thus the sum of the two series diverges as well.

Recall the Monotonic Sequence Theorem: If a non-decreasing sequence is bounded from above, then it is convergent. A consequence of this is the following theorem.

Theorem: A series  $\sum_{n=1}^{\infty} a_n$  of non-negative terms converges if and only if its partial sums are bounded from above.

Proof: If the sequence  $\{a_n\}$  has only non-negative terms, then the sequence  $\{s_n\}$  of partial sums is non-decreasing.

Example 5.) The series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

Proof: We group the terms into clusters that each add up to a number greater than  $\frac{1}{2}$ . Since there are infinitely many such clusters, the sequence of partial sums is not bounded from above. Thus the series diverges.

$$1 + \frac{1}{2} + \underbrace{\left( \frac{1}{3} + \frac{1}{4} \right)}_{> \frac{2}{4}} + \underbrace{\left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right)}_{> \frac{4}{8}} + \underbrace{\left( \frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16} \right)}_{> \frac{8}{16}} + \dots$$

This is an excellent example to demonstrate that the  $n$ th term may approach zero and yet the series diverges.

Theorem: (The Integral Test) Let  $\{a_n\}$  be a sequence of positive terms. Suppose that  $a_n = f(n)$ , where  $f$  is a continuous, positive, decreasing function of  $x$  for all  $x \geq N$  ( $N$  a positive integer). Then the series  $\sum_{n=N}^{\infty} a_n$  and the integral  $\int_N^{\infty} f(x) dx$  both converge or both diverge.

Proof. Suppose that the conditions hold. Suppose that  $N = 1$ . If the function  $f$  is decreasing, then all left-hand Riemann sums overestimate the area under the graph and all right-hand sums underestimate the same area. Consider now the following:

$a_1 + a_2 + a_3 + \dots + a_n$  is a left sum for  $f$  on the interval  $[1, n + 1]$  with the partition  $\{1, 2, 3, \dots, n + 1\}$

$$\int_1^{n+1} f(x) dx \leq a_1 + a_2 + a_3 + \dots + a_n$$

$a_2 + a_3 + \dots + a_n$  is a right sum for  $f$  on  $[1, n]$  with the partition  $\{1, 2, 3, \dots, n\}$ . Thus

$$a_2 + a_3 + \dots + a_n \leq \int_1^n f(x) dx$$

Add  $a_1$  to both sides:

$$a_1 + a_2 + a_3 + \dots + a_n \leq a_1 + \int_1^n f(x) dx$$

And so we have that

$$\int_1^{n+1} f(x) dx \leq a_1 + a_2 + a_3 + \dots + a_n \leq a_1 + \int_1^n f(x) dx$$

These inequalities are true for all  $n$ , and continue to hold as  $n$  approaches infinity.

If  $\int_1^{\infty} f(x) dx$  is finite, then  $a_1 + \int_1^{\infty} f(x) dx$  is an upper limit of the sequence of partial sums, and so  $\sum_{n=1}^{\infty} a_n$  is finite.

If  $\int_1^{\infty} f(x) dx$  is infinite, then  $\int_1^{n+1} f(x) dx$  is not bounded from above and so the sequence of partial sums is also not

bounded from above, thus  $\sum_{n=1}^{\infty} a_n$  is infinite. Thus the series and the integral are both finite or both infinite.

Example 6.) Determine whether the given series is convergent or not.

a)  $\sum_{n=1}^{\infty} \frac{1}{n}$

This will be the second time we prove that this series diverges, but it is a very famous series. It is called the harmonic series. We will use the integral test. The sequence  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$  is decreasing and all terms are positive.

Also, the function  $f(x) = \frac{1}{x}$  is continuous, positive, and decreasing on  $[1, \infty)$ , so we may apply the integral test.

Since  $\int_1^{\infty} \frac{1}{x} dx$  diverges, so does  $\sum_{n=1}^{\infty} \frac{1}{n}$ .

b)  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

We will use the integral test. The sequence  $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$  is decreasing and all terms are positive. Also, the function  $f(x) = \frac{1}{x^2}$  is continuous, positive, and decreasing on  $[1, \infty)$ , so we may apply the integral test. Since

$\int_1^{\infty} \frac{1}{x^2} dx$  converges, so does  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .

c)  $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$

We will use the integral test. The sequence  $\frac{1}{3 \ln 3}, \frac{1}{4 \ln 4}, \frac{1}{5 \ln 5}, \dots$  is decreasing and all terms are positive. Also, the function  $f(x) = \frac{1}{x \ln x}$  is continuous, and positive on  $[3, \infty)$ . We may apply the integral test, but first we must prove that  $f(x) = \frac{1}{x \ln x}$  is decreasing on  $[3, \infty)$ . We differentiate  $f(x)$

$$\frac{d}{dx} \left( \frac{1}{x \ln x} \right) = \frac{d}{dx} \left( (x \ln x)^{-1} \right) = -1 (x \ln x)^{-2} (\ln x + 1) = -\frac{1 + \ln x}{(x \ln x)^2}$$

Since the derivative of  $f$  is negative on  $[3, \infty)$ , the function is decreasing and so the conditions for applying the integral test all hold. We compute the improper integral  $\int_3^{\infty} \frac{1}{x \ln x} dx$ . We use integration by substitution to compute the indefinite integral: let  $u = \ln x$  then  $du = \frac{1}{x} dx$

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x} \left( \frac{1}{x} dx \right) = \int \frac{1}{u} du = \ln |u| + C = \ln |\ln x| + C$$

Now for the improper integral:

$$\int_3^{\infty} \frac{1}{x \ln x} dx = \lim_{N \rightarrow \infty} \int_3^N \frac{1}{x \ln x} dx = \lim_{N \rightarrow \infty} \ln |\ln x| \Big|_3^N = \lim_{N \rightarrow \infty} (\ln \ln N - \ln \ln 3) = \infty$$

Since the integral diverges, so does the series  $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$ .

## Practice Problems

Determine which of the following series converge and which diverge.

Please note that most of these problems can be solved using several different methods.

1. 
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}$$

6. 
$$\sum_{n=1}^{\infty} \sqrt[n]{n}$$

11. 
$$\sum_{n=1}^{\infty} \operatorname{sech} n$$

2. 
$$\sum_{n=1}^{\infty} \frac{2n + 1}{n^2 (n + 1)^2}$$

7. 
$$\sum_{n=1}^{\infty} \frac{8}{n^2}$$

12. 
$$\sum_{n=2}^{\infty} \frac{n + 2}{n^2 - n}$$

3. 
$$\sum_{n=0}^{\infty} \frac{1}{n^2 + 1}$$

8. 
$$\sum_{n=2}^{\infty} \frac{\ln n}{n}$$

13. 
$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

4. 
$$\sum_{n=0}^{\infty} \frac{5^n + 3^n}{4^n}$$

9. 
$$\sum_{n=1}^{\infty} \frac{1}{1 + n^2}$$

14. 
$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

5. 
$$\sum_{n=0}^{\infty} \sqrt{n}$$

10. 
$$\sum_{n=1}^{\infty} \frac{2}{e^n}$$

## Answers - Practice Problems

- 1.) converges to  $\frac{3}{4}$     2.) converges to 1    3.) converges    4.) diverges    5.) diverges    6.) diverges  
 7.) converges    8.) diverges    9.) converges    10.) converges    11.) converges    12.) diverges  
 13.) converges    14.) diverges

## Solutions

1.  $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}$  converges to  $\frac{3}{4}$ , it is a telescoping sum

$$a_n = \frac{1}{n^2 + 2n} = \frac{1}{n(n+2)} = \frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2} = \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right)$$

$$s_1 = a_1 = \frac{1}{2} \left( \frac{1}{1} - \frac{1}{3} \right) = \frac{1}{3}$$

$$s_2 = a_1 + a_2 = \frac{1}{2} \left[ \left( \frac{1}{1} - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) \right] = \frac{11}{24}$$

$$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} \left[ \left( \frac{1}{1} - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) \right] = \frac{1}{2} \left( 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} \right) = \frac{21}{40}$$

$$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} \left[ \left( \frac{1}{1} - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{4} - \frac{1}{6} \right) \right] = \frac{1}{2} \left( 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6} \right) = \frac{17}{30}$$

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n \\ &= \frac{1}{2} \left[ \left( \frac{1}{1} - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \dots + \left( \frac{1}{n-1} - \frac{1}{n+1} \right) + \left( \frac{1}{n} - \frac{1}{n+2} \right) \right] \\ &= \frac{1}{2} \left( 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{1}{2} \left( \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \end{aligned}$$

Thus  $s_n = \frac{1}{2} \left( \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right)$  and so

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{3}{4}$$

2.  $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$  converges to 1

Solution: We use partial fractions to decompose  $a_n = \frac{2n+1}{n^2(n+1)^2}$

$$\begin{aligned} \frac{2n+1}{n^2(n+1)^2} &= \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2} \\ \frac{2n+1}{n^2(n+1)^2} &= \frac{An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2}{n^2(n+1)^2} \\ 2n+1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 \end{aligned}$$



Let  $n = 0$

$$\begin{aligned} 2n + 1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 && \text{becomes} \\ 1 &= B \end{aligned}$$

Let  $n = -1$

$$\begin{aligned} 2n + 1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 && \text{becomes} \\ -1 &= D \end{aligned}$$

Let  $n = 1$

$$\begin{aligned} 2n + 1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 && \text{becomes} \\ 3 &= 4A + 4 + 2C - 1 \\ 0 &= 4A - 2C \\ C &= 2A \end{aligned}$$

Let  $n = 2$

$$\begin{aligned} 2n + 1 &= An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 && \text{becomes} \\ 5 &= 18A + 9 + 12C - 4 \\ 0 &= 18A + 12C \\ 0 &= 3A + 2C \end{aligned}$$

So we finally have this system:

$$\begin{aligned} C &= 2A \\ 3A + 2C &= 0 \end{aligned}$$

we solve the system and obtain  $A = C = 0$ . Thus the decomposition of  $a_n$  is as follows:

$$a_n = \frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

Thus the sequence of partial sums is as follows:

$$\begin{aligned} s_1 &= a_1 = \frac{1}{1^2} - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4} \\ s_2 &= a_1 + a_2 = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) = 1 - \frac{1}{9} = \frac{8}{9} \\ s_3 &= a_1 + a_2 + a_3 = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) = 1 - \frac{1}{16} = \frac{15}{16} \\ s_n &= a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n \\ &= \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \dots + \left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right) + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \\ &= 1 - \frac{1}{(n+1)^2} \end{aligned}$$

3.  $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$  converges by integral test

4.  $\sum_{n=0}^{\infty} \frac{5^n + 3^n}{4^n}$  diverges because

$$\sum_{n=0}^{\infty} \frac{5^n + 3^n}{4^n} = \sum_{n=0}^{\infty} \frac{5^n}{4^n} + \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$$

and the first geometric series diverges

5.  $\sum_{n=0}^{\infty} \sqrt{n}$  diverges since  $a_n$  does not approach 0

6.  $\sum_{n=1}^{\infty} \sqrt[n]{n}$  diverges since  $a_n$  does not approach 0

7.  $\sum_{n=1}^{\infty} \frac{8}{n^2}$  converges since it is a constant times  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

8.  $\sum_{n=2}^{\infty} \frac{\ln n}{n}$  diverges by the integral test:

The function  $f(x) = \frac{\ln x}{x}$  is continuous and positive on  $[2, \infty)$ . We may apply the integral test if we also show that  $f$  is decreasing there.  $\frac{d}{dx} \left(\frac{\ln x}{x}\right) = \frac{1}{x^2} (1 - \ln x)$  which is negative on  $[3, \infty)$ . So after  $n = 3$ , the conditions hold.  $\int \frac{\ln x}{x} dx = \frac{1}{2} (\ln x)^2 + C$ . The improper integral  $\int_3^{\infty} \frac{\ln x}{x} dx$  diverges and therefore so does the series.

9.  $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$  converges by the integral test

10.  $\sum_{n=1}^{\infty} \frac{2}{e^n}$  converges, a geometric sequence with  $a = \frac{2}{e}$  and  $r = \frac{1}{e}$  with  $0 < \frac{1}{e} < 1$

11.  $\sum_{n=1}^{\infty} \operatorname{sech} n$  converges by the integral test

$$\frac{d}{dx} \left( \frac{2}{e^x + e^{-x}} \right) = 2 \frac{d}{dx} (e^x + e^{-x})^{-1} = 2(-1)(e^x + e^{-x})^{-2} (e^x - e^{-x})$$

$$\int \frac{2}{e^x + e^{-x}} dx = 2 \arctan(e^x) + C \quad \int_0^{\infty} \frac{2}{e^x + e^{-x}} dx = \frac{\pi}{2}$$

12.  $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$  diverges

We decompose  $a_n$  using partial fractions:

$$\frac{n+2}{n^2-n} = \frac{3}{n-1} - \frac{2}{n}$$

Then  $s_n$  becomes  $s_n = 2 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right)$

$$\begin{aligned}
 s_n &= \frac{3}{1} - \underbrace{\frac{2}{2} + \frac{3}{2}} + \underbrace{\frac{2}{3} + \frac{3}{3}} - \underbrace{\frac{2}{4} + \frac{3}{4}} - \frac{2}{5} + \dots - \underbrace{\frac{2}{n-1} + \frac{3}{n-1}} - \frac{2}{n} \\
 &= 3 + \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n-1} \right) - \frac{2}{n}
 \end{aligned}$$

This shows that the sequence of partial sums is not bounded from above, thus the series diverges.

13.  $\sum_{n=1}^{\infty} \frac{1}{n^3}$  converges by the integral test

14.  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$  diverges by the integral test

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