The Ratio and Root Tests

Theorem: (The Ratio Test) Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\rho$$

Case1: If $\rho<1$ then the series converges.
Case 2: If $\rho>1$ or $\rho=\infty$ then the series diverges.
Case 3: If $\rho=1$, the test is inconclusive and other methods need to be used to determine convergence.

proof: Suppose that the conditions hold.

Case 1: $\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\rho$ and $\rho<1$. We will show that then $\sum a_n$ is less than a convergent geometric series (comparison test).

Let r be a positive number with $\rho<r<1$. Let $\varepsilon=r-\rho$. Since $\dfrac{a_{n+1}}{a_n}$ converges to ρ, there exists N such that for all $n\in\mathbb{N}$, if $n>N$, then

$$\rho-\varepsilon<\frac{a_{n+1}}{a_n}<\rho+\varepsilon$$

$$\begin{aligned}\frac{a_{n+1}}{a_n}&<\rho+\varepsilon\\ \frac{a_{n+1}}{a_n}&<\rho+r-\rho=r\\ a_{n+1}&<ra_n\end{aligned}$$

This means that

$$\begin{aligned}a_{N+1}&<ra_N\\ a_{N+2}&<ra_{N+1}<r^2a_N\\ a_{N+3}&<ra_{N+2}<r^3a_N\\ &\vdots\\ a_{N+m}&<ra_{N+m-1}<r^ma_N\end{aligned}$$

Define $\sum c_n$ as follows. For $n=1,2,...,N$ let $c_n=a_n$. For $n\geq N$, let $c_N=a_N$, $c_{N+1}=ra_N$, $c_{N+2}=r^2a_N$, ..., $c_{N+m}=r^ma_N$. This is (eventually) a geometric series with $0<r<1$ and so it converges. Now for all n, $a_n\leq c_n$ and so $\sum a_n$ also converges by the comparison theorem.

Case 2: Suppose that $\rho>1$ or is infinite. Then there exists N so that for all $n>N$, $\dfrac{a_{n+1}}{a_n}>1$ which means that $a_{n+1}>a_n$ and so the terms of the sequence increase and thus fail to approach zero. The series diverges by the nth Term Test.

Case 3: Suppose that $\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1$. Consider now the series $\sum\frac{1}{n}$ and $\sum\frac{1}{n^2}$. In case of $\sum\frac{1}{n}$,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{n+1}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=1$$

and in the case of $\sum \frac{1}{n^2}$,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = 1$$

The ratio test gives us the same limit for the ratio of consecutive terms, yet one series is convegrent and the other is divergent. So, if the ratio of the consecutive terms approach 1, other tests of convergence must be used. ■

Theorem: (The Root Test) Let $\sum a_n$ be a series with $a_n \geq 0$ for all $n \geq N$ (i.e. with eventually non-negative terms) and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

Case1: If $\rho < 1$ then the series converges.
Case 2: If $\rho > 1$ or $\rho = \infty$ then the series diverges.
Case 3: If $\rho = 1$, the test is inconclusive and other methods need to be used to determine convergence.

proof: Case 1. Suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$. Let $\varepsilon > 0$ be chosen small enough so that $\rho + \varepsilon < 1$. There exists N so that for all $n \geq N$, $a_n \geq 0$. Also, there exists M so that for all $n \geq M$, $\sqrt[n]{a_n} < \rho + \varepsilon$. Define $K = \max(N, M)$. So, for all $n \geq K$, we have that

$$\sqrt[n]{a_n} < \rho + \varepsilon \implies a_n < (\rho+\varepsilon)^n$$

The geometric sequence $\sum_{n=K}^{\infty} (\rho+\varepsilon)^n$ is convergent since $\rho + \varepsilon < 1$. By the comparison test, $\sum a_n$ converges.

Case 2. If $\rho > 1$ or infinite, then there exists M so that if $n \geq M$, $\sqrt[n]{a_n} > 1$ so that $a_n > 1$. This means that the sequence fails to converge to zero, and so the series diverges by the nth term test.

Case 3: Suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$. Consider now the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. In case of $\sum \frac{1}{n}$,

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \lim_{x\to\infty} \frac{1}{\sqrt[x]{x}} = \lim_{x\to\infty} x^{-1/x} = \lim_{x\to\infty} e^{\ln x^{-1/x}} = \lim_{x\to\infty} e^{-(1/x)\ln x} \\ &= e^{\lim_{x\to\infty} -(1/x)\ln x} = \frac{1}{e^{\lim_{x\to\infty}(\ln x/x)}} = \frac{1}{e^{\lim_{x\to\infty}(1/x)}} = \frac{1}{e^0} = 1\end{aligned}$$

and in the case of $\sum \frac{1}{n^2}$,

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n^2}} = \lim_{x\to\infty} \frac{1}{\sqrt[x]{x^2}} = \lim_{x\to\infty} \left(x^2\right)^{-1/x} = \lim_{x\to\infty} e^{\ln x^{-2/x}} = \lim_{x\to\infty} e^{-(2/x)\ln x} \\ &= e^{\lim_{x\to\infty} -(2/x)\ln x} = \frac{1}{e^{\lim_{x\to\infty}(2\ln x/x)}} = \frac{1}{e^{\lim_{x\to\infty}(2/x)}} = \frac{1}{e^0} = 1\end{aligned}$$

The root test gives us the same limit for the nth root of terms, yet one series is convegrent and the other is divergent. So, if the nth root of the terms approach 1, other tests of convergence must be used. ■

Sample Problems

1. In case of the following series, use the ratio test to determine convergence or divergence of the series.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{(2n)!}$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{3n-1}+1}{3^n}$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n-1}{5^n}$

e) $\displaystyle\sum_{n=0}^{\infty} n!\ 2^{1-n}$

f) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n-1}}{(3n+4)\,5^n}$

g) $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3}$

h) $\displaystyle\sum_{n=1}^{\infty} n^{100}2^{-2n}$

2. In case of the following series, use the root test to determine convergence or divergence of the series.

a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{3n}\right)^n$

b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

c) $\displaystyle\sum_{n=1}^{\infty} 5^n 2^{-2n}$

d) $\displaystyle\sum_{n=1}^{\infty} n^{100}2^{-2n}$

3. Use any method we know so far (that is: direct limit of s_n, comparison test, integral test, ratio test, root test, nth term test, or grouping of the terms) to determine whether the following series converge or diverge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{3n+1}\right)^{2n+1}$

d) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{7^n}$

f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{n+1}}$

g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

h) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$

i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

j) $\displaystyle\sum_{n=1}^{\infty} \frac{n\ 2^n\,(n+1)!}{3^n n!}$

Sample Problems - Solutions

1. In case of the following series, use the ratio test to determine convergence or divergence of the series.

a) $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{3^n}$ diverges by the ratio test

$$\rho=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\dfrac{(n+1)!}{3^{n+1}}}{\dfrac{n!}{3^n}}=\lim_{n\to\infty}\left(\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!}\right)=\lim_{n\to\infty}\left(\frac{3^n(n+1)n!}{3\cdot 3^n n!}\right)=\lim_{n\to\infty}\frac{(n+1)}{3}=\infty$$

b) $\displaystyle\sum_{n=0}^{\infty}\frac{5^n}{(2n)!}$ converges by the ratio test

$$\begin{aligned}\rho&=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\dfrac{5^{n+1}}{(2(n+1))!}}{\dfrac{5^n}{(2n)!}}=\lim_{n\to\infty}\left(\frac{5\cdot 5^n}{(2n+2)!}\cdot\frac{(2n)!}{5^n}\right)=\lim_{n\to\infty}\frac{5(2n)!}{(2n+2)(2n+1)(2n)!}\\&=\lim_{n\to\infty}\frac{5}{(2n+2)(2n+1)}=0\end{aligned}$$

c) $\displaystyle\sum_{n=0}^{\infty}\frac{2^{3n-1}+1}{3^n}$ diverges by the ratio test

$$\begin{aligned}\rho&=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\dfrac{2^{3(n+1)-1}+1}{3^{n+1}}}{\dfrac{2^{3n-1}+1}{3^n}}=\lim_{n\to\infty}\left(\frac{2^{3n+3-1}+1}{3\cdot 3^n}\cdot\frac{3^n}{2^{3n-1}+1}\right)=\lim_{n\to\infty}\frac{2^{3n+2}+1}{3\left(2^{3n-1}+1\right)}\\&=\frac{1}{3}\lim_{n\to\infty}\frac{4\cdot 2^{3n}+1}{\frac{1}{2}\cdot 2^{3n}+1}=\frac{1}{3}\lim_{n\to\infty}\frac{4\left(2^{3n}+\frac{1}{4}\right)}{\frac{1}{2}\left(2^{3n}+\frac{1}{2}\right)}=\frac{1}{3}\lim_{n\to\infty}\frac{8\left(8^n+\frac{1}{4}\right)}{8^n+\frac{1}{2}}=\frac{8}{3}\lim_{n\to\infty}\frac{8^n+\frac{1}{4}}{8^n+\frac{1}{2}}\\&=\frac{8}{3}\lim_{n\to\infty}\frac{8^n\left(1+\frac{1}{4\cdot 8^n}\right)}{8^n\left(1+\frac{1}{2\cdot 8^n}\right)}=\frac{8}{3}\lim_{n\to\infty}\frac{1+\frac{1}{4\cdot 8^n}}{1+\frac{1}{2\cdot 8^n}}=\frac{8}{3}\cdot 1=\frac{8}{3}>1\end{aligned}$$

d) $\displaystyle\sum_{n=0}^{\infty}\frac{3^n-1}{5^n}$ converges by the ratio test

$$\begin{aligned}\rho&=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\dfrac{3^{n+1}-1}{5^{n+1}}}{\dfrac{3^n-1}{5^n}}=\lim_{n\to\infty}\left(\frac{3^{n+1}-1}{5^{n+1}}\cdot\frac{5^n}{3^n-1}\right)=\lim_{n\to\infty}\left(\frac{3\cdot 3^n-1}{5\cdot 5^n}\cdot\frac{5^n}{3^n-1}\right)\\&=\lim_{n\to\infty}\left(\frac{3\cdot 3^n-1}{5\left(3^n-1\right)}\right)=\lim_{n\to\infty}\left(\frac{3\left(3^n-\frac{1}{3}\right)}{5\left(3^n-1\right)}\right)=\frac{3}{5}\lim_{n\to\infty}\frac{3^n-\frac{1}{3}}{3^n-1}=\frac{3}{5}\lim_{n\to\infty}\frac{3^n\left(1-\frac{1}{3\cdot 3^n}\right)}{3^n\left(1-\frac{1}{3^n}\right)}\\&=\frac{3}{5}\lim_{n\to\infty}\frac{1-\frac{1}{3\cdot 3^n}}{1-\frac{1}{3^n}}=\frac{3}{5}\cdot 1=\frac{3}{5}<1\end{aligned}$$

e) $\displaystyle\sum_{n=0}^{\infty} n!\ 2^{1-n}$ diverges by the ratio test

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\ 2^{1-(n+1)}}{n!\ 2^{1-n}} = \lim_{n\to\infty} \frac{(n+1)\,n!\ \dfrac{2}{2^{n+1}}}{n!\ \dfrac{2}{2^n}} = \lim_{n\to\infty} \frac{(n+1)\,n!}{n!} \cdot \frac{2}{2^{n+1}} \cdot \frac{2^n}{2}$$

$$= \lim_{n\to\infty} (n+1) \cdot \frac{1}{2\cdot 2^n} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{n+1}{2} = \infty$$

f) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n-1}}{(3n+4)\,5^n}$ converges by the ratio test

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{3^{n+1-1}}{(3\,(n+1)+4)\,5^{n+1}}}{\dfrac{3^{n-1}}{(3n+4)\,5^n}} = \lim_{n\to\infty} \left(\frac{3^{n+1-1}}{(3\,(n+1)+4)\,5^{n+1}} \cdot \frac{(3n+4)\,5^n}{3^{n-1}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{3^n}{(3n+3+4)\,5\cdot 5^n} \cdot \frac{(3n+4)\,5^n}{\frac{1}{3}\cdot 3^n} \right) = \lim_{n\to\infty} \frac{3\,(3n+4)}{5\,(3n+7)} = \lim_{n\to\infty} \frac{3\cdot 3n\left(1+\dfrac{4}{3n}\right)}{5\cdot 3n\left(1+\dfrac{7}{3n}\right)}$$

$$= \frac{3}{5} \lim_{n\to\infty} \frac{1+\dfrac{4}{3n}}{1+\dfrac{7}{3n}} = \frac{3}{5}\cdot 1 = \frac{3}{5} < 1$$

g) $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3}$ diverges by the ratio test

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(3\,(n+1))!}{((n+1)!)^3}}{\dfrac{(3n)!}{(n!)^3}} = \lim_{n\to\infty} \left(\frac{(3\,(n+1))!}{((n+1)!)^3} \cdot \frac{(3n)!}{(n!)^3} \right) = \lim_{n\to\infty} \left(\frac{(3n+3)!}{((n+1)!)^3} \cdot \frac{(n!)^3}{(3n)!} \right)$$

$$= \lim_{n\to\infty} \left(\frac{(3n+3)\,(3n+2)\,(3n+1)\,(3n)!}{((n+1)\,n!)^3} \cdot \frac{(n!)^3}{(3n)!} \right) = \lim_{n\to\infty} \frac{(3n+3)\,(3n+2)\,(3n+1)\,(n!)^3}{(n+1)^3\,(n!)^3}$$

$$= \lim_{n\to\infty} \frac{3\,(n+1)\,(3n+2)\,(3n+1)}{(n+1)^3} = 3\lim_{n\to\infty} \frac{(3n+2)\,(3n+1)}{(n+1)^2} = 3\lim_{n\to\infty} \frac{9n^2+9n+2}{n^2+2n+1}$$

$$= 3\lim_{n\to\infty} \frac{9n^2\left(1+\dfrac{1}{n}+\dfrac{2}{9n^2}\right)}{n^2\left(1+\dfrac{2}{n}+\dfrac{1}{n^2}\right)} = 27\lim_{n\to\infty} \frac{1+\dfrac{1}{n}+\dfrac{2}{9n^2}}{1+\dfrac{2}{n}+\dfrac{1}{n^2}} = 27\cdot 1 = 27 > 1$$

h) $\displaystyle\sum_{n=1}^{\infty} n^{100}2^{-2n}$ converges by the ratio test

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{100}\,2^{-2(n+1)}}{n^{100}2^{-2n}} = \lim_{n\to\infty} \frac{(n+1)^{100}\,2^{-2n-2}}{n^{100}2^{-2n}} = \lim_{n\to\infty} \frac{(n+1)^{100}\,2^{-2n}\left(\frac{1}{4}\right)}{n^{100}2^{-2n}}$$

$$= \frac{1}{4}\lim_{n\to\infty} \frac{(n+1)^{100}}{n^{100}} = \frac{1}{4}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{100} = \frac{1}{4}\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^{100} = \frac{1}{4}\left(\lim_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{n}\right)^{100}$$

$$= \frac{1}{4}\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)^{100} = \frac{1}{4}\cdot 1 = \frac{1}{4} < 1$$

2. In case of the following series, use the root test to determine convergence or divergence of the series.

a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{3n}\right)^n$ converges by the root test

$$\rho = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n+1}{3n}\right)^n} = \lim_{n\to\infty}\frac{n+1}{3n} = \lim_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{3n} = \frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{1}{3}\cdot 1 = \frac{1}{3} < 1$$

b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{1+n}\right)^n$ converges by the root test

$$\rho = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \lim_{n\to\infty}\frac{1}{1+n} = 0 < 1$$

c) $\displaystyle\sum_{n=1}^{\infty} 5^n 2^{-2n}$ diverges by the root test

$$\rho = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{5^n 2^{-2n}} = \lim_{n\to\infty}\sqrt[n]{\frac{5^n}{4^n}} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{5}{4}\right)^n} = \frac{5}{4} > 1$$

d) $\displaystyle\sum_{n=1}^{\infty} n^{100}2^{-2n}$ converges by the root test

$$\rho = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{n^{100}2^{-2n}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^{100}}{4^n}} = \lim_{n\to\infty}\frac{1}{4}\sqrt[n]{n^{100}} = \frac{1}{4}\lim_{n\to\infty} n^{100/n} = \frac{1}{4}\lim_{n\to\infty}\left(n^{1/n}\right)^{100}$$

$$= \frac{1}{4}\left(\lim_{n\to\infty} n^{1/n}\right)^{100} = \frac{1}{4}\cdot 1^{100} = \frac{1}{4}\cdot 1 = \frac{1}{4} < 1$$

3. Use any method we know so far (that is: direct limit of s_n, comparison test, integral test, ratio test, root test, nth term test, or grouping of the terms) to determine whether the following series converge or diverge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ converges by the ratio test

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)^{10}}{10^{n+1}}}{\dfrac{n^{10}}{10^n}} = \lim_{n\to\infty}\left(\frac{(n+1)^{10}}{10^{n+1}}\cdot\frac{10^n}{n^{10}}\right) = \lim_{n\to\infty}\frac{10^n\ (n+1)^{10}}{10\cdot 10^n\ n^{10}} = \lim_{n\to\infty}\frac{(n+1)^{10}}{10n^{10}} \\
&= \frac{1}{10}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{10} = \frac{1}{10}\lim_{n\to\infty}\left(\frac{1+\dfrac{1}{n}}{1}\right)^{10} = \frac{1}{10}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{10} = \frac{1}{10}\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)^{10} \\
&= \frac{1}{10}\cdot 1^{10} = \frac{1}{10} < 1
\end{aligned}$$

The root test also works:

$$\rho = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{n^{10}}{10^n}} = \lim_{n\to\infty}\frac{1}{10}\sqrt[n]{n^{10}} = \frac{1}{10}\lim_{n\to\infty}\left(\sqrt[n]{n}\right)^{10} = \frac{1}{10}\left(\lim_{n\to\infty}\sqrt[n]{n}\right)^{10} = \frac{1}{10}\cdot 1^{10} = \frac{1}{10} < 1$$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by the ratio test

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\dfrac{(n+1)!}{(n+1)^{n+1}}}{\dfrac{n!}{n^n}} = \lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right) = \lim_{n\to\infty}\frac{(n+1)\,n!n^n}{(n+1)^{n+1}\,n!} = \lim_{n\to\infty}\frac{(n+1)\,n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty}\frac{(n+1)\,n^n}{(n+1)\,(n+1)^n} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^{-1}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n} \\
&= \left(\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\right)^{-1} = \frac{1}{\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1
\end{aligned}$$

c) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{3n+1}\right)^{2n+1}$ converges by the root test

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n+1}{3n+1}\right)^{2n+1}} = \lim_{n\to\infty}\left(\frac{n+1}{3n+1}\right)^{(2n+1)/n} = \lim_{n\to\infty} e^{\ln (n+1)/(3n+1)^{(2n+1)/n}} \\
&= \lim_{n\to\infty} e^{[(2n+1)/n]\cdot\ln[(n+1)/(3n+1)]} = e^{\lim\limits_{n\to\infty}[(2n+1)/n]\cdot\ln[(n+1)/(3n+1)]}
\end{aligned}$$

The exponent is

$$\begin{aligned}
X &= \lim_{n\to\infty}\left(\frac{2n+1}{n}\cdot\ln\frac{n+1}{3n+1}\right) = \lim_{n\to\infty}\left(\left(2+\frac{1}{n}\right)\cdot\ln\frac{n+1}{3n+1}\right) = \lim_{n\to\infty}\left(\left(2+\frac{1}{n}\right)\cdot\lim_{n\to\infty}\ln\frac{n+1}{3n+1}\right) \\
&= 2\lim_{n\to\infty}\ln\frac{n+1}{3n+1} = 2\lim_{n\to\infty}\ln\frac{1+\dfrac{1}{n}}{3+\dfrac{1}{n}} = 2\ln\frac{1}{3} = \ln\frac{1}{9}
\end{aligned}$$

and so

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = e^{\ln(1/9)} = \frac{1}{9} < 1$$

d) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$ converges by the ratio test

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2 e^{-(n+1)}}{n^2 e^{-n}} = \lim_{n\to\infty} \frac{(n+1)^2 e^{-n-1}}{n^2 e^{-n}} = \lim_{n\to\infty} \frac{(n+1)^2 e^{-n}\left(\frac{1}{e}\right)}{n^2 e^{-n}} = \frac{1}{e}\lim_{n\to\infty} \frac{(n+1)^2}{n^2} \\ &= \frac{1}{e}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = \frac{1}{e}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = \frac{1}{e} < 1\end{aligned}$$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{7^n}$ converges because it is a geometric series with $r = -\frac{2}{7}$ and $-1 < r < 1$.

f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{n+1}}$ converges by the root test

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^{n+1}}} = \lim_{n\to\infty} n^{-(n+1)/n} = \lim_{n\to\infty} e^{\ln n^{-(n+1)/n}} = e^{\lim\limits_{n\to\infty}\left(\ln n^{-(n+1)/n}\right)}$$

The exponent is

$$\lim_{n\to\infty}\left(\ln n^{-(n+1)/n}\right) = \lim_{n\to\infty}\left(-\frac{n+1}{n}\ln n\right) = \lim_{n\to\infty}\left(-\frac{n+1}{n}\right)\cdot\lim_{n\to\infty}\ln n = -1\cdot\infty = -\infty$$

thus

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{x\to-\infty} e^x = 0 < 1$$

g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$ converges by the root test

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n} = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n^2}\right) = 0$$

h) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$ converges by the ratio test

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{((n+1)+1)((n+1)+2)}{(n+1)!}}{\dfrac{(n+1)(n+2)}{n!}} = \lim_{n\to\infty}\left(\frac{(n+2)(n+3)}{(n+1)!}\cdot\frac{n!}{(n+1)(n+2)}\right) \\ &= \lim_{n\to\infty}\left(\frac{(n+2)(n+3)}{(n+1)\,n!}\cdot\frac{n!}{(n+1)(n+2)}\right) = \lim_{n\to\infty}\frac{n+3}{(n+1)^2} = \lim_{n\to\infty}\frac{n+3}{n^2+2n+1} = \lim_{n\to\infty}\frac{n\left(1+\frac{3}{n}\right)}{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)} \\ &= \lim_{n\to\infty}\frac{n}{n^2}\cdot\lim_{n\to\infty}\frac{1+\frac{3}{n}}{1+\frac{2}{n}+\frac{1}{n^2}} = 0\cdot 1 = 0\end{aligned}$$

i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$ diverges by the root test

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(2^n)^2}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{4^n}} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{4}\right)^n} = \lim_{n\to\infty} \frac{n}{4} = \infty$$

j) $\displaystyle\sum_{n=1}^{\infty} \frac{n\ 2^n\,(n+1)!}{3^n n!}$ converges by the ratio test

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)\ \ 2^{n+1}\,((n+1)+1)!}{3^{n+1}\,(n+1)!}}{\dfrac{n\ 2^n\,(n+1)!}{3^n n!}} = \lim_{n\to\infty} \left(\frac{(n+1)\ \ 2^{n+1}\,(n+2)!}{3^{n+1}\,(n+1)!} \cdot \frac{3^n n!}{n\ 2^n\,(n+1)!} \right) \\
&= \lim_{n\to\infty} \left(\frac{(n+1)\ \ 2\cdot 2^n\,(n+2)\,(n+1)!}{3\cdot 3^n\,(n+1)!} \cdot \frac{3^n n!}{n\ 2^n\,(n+1)!} \right) = \lim_{n\to\infty} \left(\frac{(n+1)\ \ 2\,(n+2)}{3} \cdot \frac{n!}{n\ \ (n+1)!} \right) \\
&= \lim_{n\to\infty} \left(\frac{(n+1)\ \ 2\,(n+2)}{3} \cdot \frac{n!}{n\ \ (n+1)\,n!} \right) = \lim_{n\to\infty} \left(\frac{(n+1)\ \ 2\,(n+2)}{3} \cdot \frac{1}{n\ \ (n+1)} \right) \\
&= \lim_{n\to\infty} \left(\frac{2}{3} \cdot \frac{n+2}{n} \right) = \frac{2}{3} \lim_{n\to\infty} \left(1 + \frac{2}{n} \right) = \frac{2}{3} < 1
\end{aligned}$$

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