

The Fundamental Theorem

Theorem: (**Fundamental Theorem of Calculus, Part 1**) If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$, differentiable on (a, b) , and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Theorem: (**Fundamental Theorem of Calculus, Part 2**) If f is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

For all positive x , define $f(x) = \int_1^x \frac{1}{t} dt$

1. f is continuous, differentiable, with $f'(x) = \frac{1}{x}$
2. $f' > 0 \implies f$ is strictly increasing $\implies$ one-to-one
3. $f(1) = 0$ If $x > 1$, then $f(x)$ is positive and if $0 < x < 1$, then $f(x)$ is negative.

This is because all area is above x -axis

4. $f(xy) = f(x) + f(y)$

Fix $y > 0$ and define $g(x) = f(xy) - f(x)$

$$g'(x) = \frac{1}{xy} \cdot y - \frac{1}{x} = 0$$

thus $g(x)$ is a constant function

$$g(1) = f(y) - f(1) = f(y)$$

$$f(y) = f(xy) - f(x) \text{ for all } x$$

5. $f\left(\frac{1}{x}\right) = -f(x)$

$$0 = f(1) = f\left(x \cdot \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

6. $f\left(\frac{x}{y}\right) = f(x) - f(y)$

$$f\left(\frac{x}{y}\right) = f\left(x \cdot \frac{1}{y}\right) = f(x) + f\left(\frac{1}{y}\right) = f(x) + (-f(y)) = f(x) - f(y)$$

7. $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow 0^+} f(x) = -\infty \implies$
range of f is $\mathbb{R}$

by the intermediate value theorem

pf: $\ln 2 > \frac{1}{2}$ picture Then $\ln 2^{2N} = \ln 2 + \ln 2 + \dots + \ln 2 > N$ and $\ln \frac{1}{2^{2N}} < -N$

For all positive x , define $\ln(x) = \int_1^x \frac{1}{t} dt$

Define $\exp x$ to be the inverse function of $\ln x$

1. domain is $\mathbb{R}$, range is $(0, \infty)$

2. $\exp(\ln x) = x$ and $\ln(\exp x) = x$

3. $\exp(x + y) = (\exp x)(\exp y)$

pf. $(\exp x)(\exp y) = \exp(\ln((\exp x)(\exp y))) =$
 $\exp(\ln(\exp x) + \ln(\exp y)) = \exp(x + y)$

4. $\exp(x - y) = \frac{\exp x}{\exp y}$

pf. $\frac{\exp x}{\exp y} = \exp\left(\ln\left(\frac{\exp x}{\exp y}\right)\right) =$
 $\exp(\ln \exp x - \ln \exp y) = \exp(x - y)$

5. $\frac{d}{dx} \exp x = \exp x \implies \exp x$ is strictly increasing,
 one-to-one

6. $\lim_{x \rightarrow -\infty} \exp x = 0$ and $\lim_{x \rightarrow \infty} \exp x = \infty$ follows from
 inverse functions

define $e = \exp 1$

denote $\exp x$ by e^x

7) $\ln e = 1$

Define $\exp_b x = \exp(x \ln b)$

7) $\exp_e x = \exp x$