Recall the definition of continuous functions.

**Definition:** (Continuity at a point) A function $y = f(x)$ is **continuous at a number $c$** of its domain if the two-sided limit exists and $\lim_{x\to c} f(x) = f(c)$.

**Definition:** (Continuity on an interval)

(*Open Interval*) A function $y = f(x)$ is continuous on an interval $(a, b)$ if it is continuous at every $c$ in $(a, b)$.
(*Closed Interval*) A function $y = f(x)$ is continuous on an interval $[a, b]$ if it is continuous at every $c$ in $(a, b)$ and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x\to b^-} f(x) = f(b), \text{ respectively.}$$

End-points of the interval require only one-sided limits.

Another way to express continuity is to say that $\lim_{h\to 0} f(x+h) = f(x)$. Another alternative statement of continuity is $\lim_{x\to c} f(x) = f\left(\lim_{x\to c} x\right)$, so that there is a commutativity between taking the limit and taking the function values.

**Definition:** Suppose that $f$ is a function and $c$ is an interior point of its domain. If the (two-sided) limit

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

exists (and is finite) we say that $f$ is **differentiable at $c$** and denote this limit as $f'(c)$.

**Theorem:** If $f$ is differentiable at $a$, then it is continuous there.

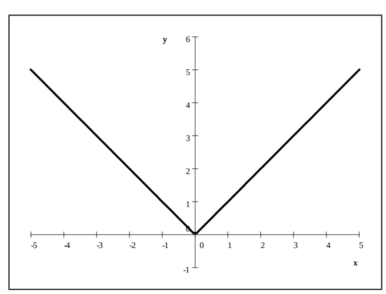
Proof: Suppose that $f$ is differentiable at a number $a$. Then $f'(a)$ exists which means that $f(a)$ exists and the limit $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ also exists and is finite. Let us start with the true statement that $0 = 0 \cdot f'(a)$.

$$\begin{aligned}
0 &= 0 \cdot f'(a) \\
0 &= \lim_{h\to 0} h \cdot \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} && \text{by the product rule of limits} \\
0 &= \lim_{h\to 0} \left(h \cdot \frac{f(a+h) - f(a)}{h}\right) && \text{cancel out } h \\
0 &= \lim_{h\to 0} \left(f(a+h) - f(a)\right) && \text{by the difference rule of limits} \\
0 &= \lim_{h\to 0} f(a+h) - \lim_{h\to 0} f(a) \\
\lim_{h\to 0} f(a) &= \lim_{h\to 0} f(a+h) && \text{by the constant rule of limits} \\
f(a) &= \lim_{h\to 0} f(a+h)
\end{aligned}$$

and $f(a) = \lim_{h\to 0} f(a+h)$ means that $f$ is continuous at $a$.

So, differentiability implies continuity. What about backwards? Are continuous functions necessarily differentiable? The answer is no. Consider the function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

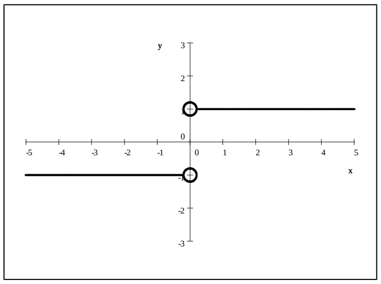


Although this function is continuous at zero, it is not differentiable there. Recall that the derivative is a two-sided limit. As $h$ approaches zero, it is negative when we compute the left-limit and positive when we compute the right limit.

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{-h-0}{h} = \lim_{h\to 0^-} -1 = -1 \quad \text{and}$$
$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h-0}{h} = \lim_{h\to 0^+} 1 = 1$$

So the derivative, a two-sided limit, is not defined at $x = 0$.

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$




When the function $f$ is continuous but not differentiable because the left-hand side derivative and the right-hand side derivative exist and are finite but not equal, the graph has a spike there. For example, the graph of $f(x) = |x|$ has a spike at $x = 0$. Differentiable functions have graphs that are continuous and smooth.