Definition: Let f be a function with domain D. Then f has a **relative maximum value** at a point c if $f(x) \leq f(c)$ for all x in D lying in some open interval containing c. A function f has a **relative minimum value** at a point c if $f(x) \geq f(c)$ for all x in D lying in some open interval containing c.

Theorem: (First Derivative Theorem for Local Extreme Values) If f is has a relative maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

Proof: Suppose that f is differentiable at c and f has a local maximum at c. Then $f'(c) = \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ exists and is a two-sided limit. For a sufficiently small positive value of h, $f(c+h)$ exists and $f(c+h) \leq f(c)$ since f has a local maximum value at c. Then $f(c+h) - f(c) \leq 0$. Divide that by positive h and get that $\dfrac{f(c+h) - f(c)}{h} \leq 0$ and so

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

Now let h be a very small negative number. Then by the same argument, $f(c+h) \leq f(c)$. Divide that by a negative h and get that $\dfrac{f(c+h) - f(c)}{h} \geq 0$ and so

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

For the two-sided limit $f'(c)$ to exists, we must have

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

Since one side is less than or equal to zero and the other is greater than or equal to zero, they both must be zero.

Definition: A **critical number** of a function f is a number c in its domain such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem: (Fermat) If f has a local maximum or minimum at c, then c is a critical number of f.

Definition: Let f be a function with domain D. Then f has an **absolute maximum value** on D at a point c if $f(x) \leq f(c)$ for all x in D and an **absolute minimum value** on D at a point c if $f(x) \geq f(c)$ for all x in D.

Theorem: (**Extreme Value Theorem**) If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

While this theorem is very important and fundamental, its proof is difficult and so it will not be covered in this class.

Closed interval method: To find absolute extrema of a continuous function f on a closed interval $[a, b]$.

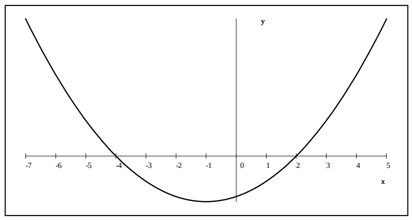
1) Find the values of f at the critical numbers of f in $[a, b]$.
2) Find the values of f at the endpoints of the interval.
3) The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example: Find all absolute and relative extrema for the function $f(x) = x^3 + 3x^2 - 24x + 24$ defined on the closed interval $[-6, 5]$.

Since this is a cubic polynomial with a positive leading coefficient, we have our initial expectations on end-behavior and one relative maximum, followed by a relative minimum. We first find these relative extrema. Since the function is dfferentiable everywhere, all critical numbers will occur where the derivative is zero.

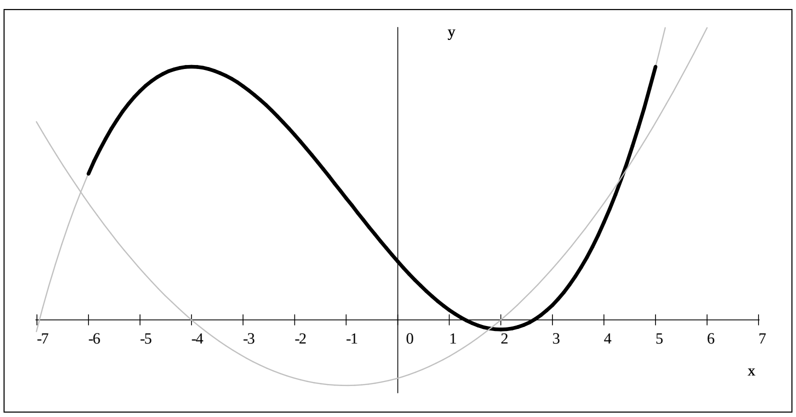
$$f'(x) = 3x^2 + 6x - 24 = 3(x+4)(x-2)$$
$$f'(x) = 0 \implies x_1 = -4 \text{ and } x_2 = 2$$

Based on the graph of f', we conclude that f' changes sign from positive to negative at $x=-4$ and from negative to positive at $x=2$. Thus f has a relative maximum at $x=-4$ and a relative minimum at $x=2$.



To find the absolute extremas, we just need to compare the function values at the critical numbers, -4 and 2 and at the endpoints of the domain, -6 and 5. We evaluate the function at these numbers and find that $f(-6) = 60$, $f(-4) = 104$, $f(2) = -4$, and $f(5) = 104$.

Based on this, f has an absolute minimum at $x=-4$ and an absolute maximum at $x=-4$ and $x=5$.



In summary:

relative minimum:	$(2, -4)$	absolute minimum:	$(2, -4)$
relative maximum:	$(-4, 104)$	absolute maximum:	$(-4, 104)$ and $(5, 104)$

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