Definition: A set S is **bounded from above** if there exists a real number U such that for all x in S, $x \leq U$.

Definition: A set S is **bounded from below** if there exists a real number L such that for all x in S, $x \geq L$.

Definition: A set S is **bounded** if it is bounded from above and from below.

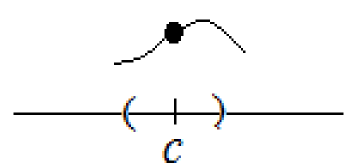
Axiom: (**Least Upper Bound Property**) Every non-empty set S of real numbers has the following property: if S is bounded from above, then there exists a least upper bound for S.

The least upper bound is also called *supremum*. The least upper bound property is a very fundamental one: it is actually the single axiom that distinguishes the set of rational numbers from the set of real numbers. Rational numbers do not have this propoerty. This property is also the key ingredient in proving the Intermediate Value Theorem.

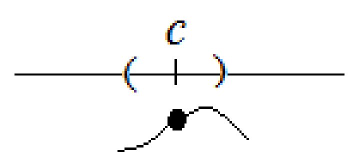
Theorem: (**The Intermediate Value Theorem**) If f is continuous on a closed interval $[a, b]$ and if $f(a) < 0$ and $f(b) > 0$, then there exists c in (a, b) so that $f(c) = 0$.

Proof: Suppose that the conditions hold. Define $S = \{x : a \leq x \leq b \text{ and } f(x) < 0\}$. This set is non-empty because a is an element of it. This set is also bounded from above, because for all x in S, $x \leq b$ and so b is an upper bound for S. By the least upper bound property, S has a least upper bound. Let us detone it by c. Since c is the least upper bound for S and b is an upper bound, we also have that $c \leq b$. Since a is in S and c is an upper bound, we also have that $a \leq c$. Thus c is in the interval $[a, b]$. We will prove that $f(c) = 0$.

We will prove that $f(c) = 0$ by showing that $f(c)$ cannot be positive or negative.



Suppose first that $f(c)$ is positive. Since f is continuous at c, that means that f is positive on some open interval containing c. That means that c is not the least upper bound for S, because any number in that interval, to the left of c is also an upper bound for S. That is impossible and so $f(c)$ cannot be positive.



Suppose now that $f(c)$ is negative. Since f is continuous at c, that means that f is negative on some open interval containing c. That means that c is not an upper bound for S, because a number in that interval, to the right of c is also an element of S. That is impossible and so $f(c)$ cannot be negative.

So $f(c) = 0$ which completes our proof. ■

Theorem: (**The Intermediate Value Theorem for Continuous Functions**) If f is continuous on a closed interval $[a, b]$ and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

Proof: Apply the Intermediate Value Theorem for $h(x) = f(x) - y_0$.■

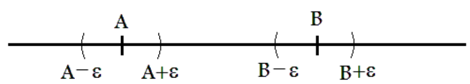
Definition: The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there exists an integer N such that for all n,

$$\text{if } n > N, \text{ then } L - \varepsilon < a_n < L + \varepsilon.$$

If no such number L exists, we say $\{a_n\}$ **diverges**.
If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ and call L the **limit** of the sequence.

Theorem: Convergent sequences have unique limits: if $\{a_n\}$ is a sequence with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} a_n = B$, then $A = B$.

Proof: Suppose for a contradiction that a sequence a_n converges to two different numbers A and B. The basic idea here is that if ε is selected to be small enough, then the ε neighborhood of A will be disjoint of the ε neighborhood of B and so a_n can not be in both intervals.



Suppose that $A \neq B$. We may assume that $A < B$. (Otherwise just re-label them so that the larger number is denoted by B.) Define $\varepsilon = \dfrac{B-A}{2}$. Since $\{a_n\}$ converges to A, there exists N_A so that for all $n > N_A$,

$$A - \varepsilon < a_n < A + \varepsilon$$

Similarly, since $\{a_n\}$ converges to B, there exists N_B so that for all $n > N_B$,

$$B - \varepsilon < a_n < B + \varepsilon$$

Now let $n > \max(N_A, N_B)$, so both conditions hold. Then

$$A - \varepsilon < a_n < A + \varepsilon \quad \text{and} \quad B - \varepsilon < a_n < B + \varepsilon$$

We will only need the right-hand side of the first inequality and the left-hand side of the other:

$$\begin{aligned}
a_n &< A + \varepsilon \quad \text{and} \quad B - \varepsilon < a_n \qquad \text{recall that } \varepsilon = \frac{B-A}{2} \\
a_n &< A + \frac{B-A}{2} \quad \text{and} \quad B - \frac{B-A}{2} < a_n \\
a_n &< \frac{2A}{2} + \frac{B-A}{2} \quad \text{and} \quad \frac{2B}{2} - \frac{B-A}{2} < a_n \\
a_n &< \frac{2A+B-A}{2} \quad \text{and} \quad \frac{2B-B+A}{2} < a_n \\
a_n &< \frac{A+B}{2} \quad \text{and} \quad \frac{A+B}{2} < a_n
\end{aligned}$$

These two can not be true at the same time. This is a contradiction, so $A \neq B$ is impossible. This completes our proof.■

Theorem: If a sequence $\{a_n\}$ is bounded from above and increasing, then it is also convergent. (Similarly, if a sequence is bounded from below and decreasing, then it is convergent.)

proof: Suppose that $\{a_n\}$ is bounded and increasing. Let L be the least upper bound for the sequence. Since L is an upper bound, $a_n \leq L$ for all n.

Let $\varepsilon > 0$ be given. Since L is the lowest upper bound, $L - \varepsilon$ is NOT an upper bound. This means that there exists m natural number such that $a_m > L - \varepsilon$. Since a_n is increasing, all subsequent terms will have this property, i.e. for all $n > m$, $a_n \geq a_m > L - \varepsilon$. thus we have that for all $n > m$

$$L - \varepsilon < a_n \leq L < L + \varepsilon$$

and so $L - \varepsilon < a_n < L + \varepsilon$ and so a_n converges to L. The proof for decreasing sequences is similar.■

Applications of the Intermediate Value Theorem

1. Prove that the function $f(x) = x^5 - 3x^4 + 8x^3 - x - 2$ has at least one zero in the interval $[0, 1]$.
2. Prove that the function $f(x) = 6x^4 + x^3 - 25x^2 - 4x + 4$ has at least two zeroes in the interval $[-1, 1]$.
3. Prove that all polynomials with an odd degree have at least one zero.

Solutions

1. f is continuous on $[0, 1]$, $f(0) = -2$, and $f(1) = 3$. By the Intermediate Value Theorem, f has a zero in $[0, 1]$.
2. f is continuous on $[-1, 1]$. $f(-1) = -12$ and $f(0) = 4$. By the Intermediate Value Theorem, f has a zero in $[-1, 0]$. Also, $f(0) = 4$ and $f(1) = -18$. By the Intermediate Value Theorem, f has a zero in $[0, 1]$.
3. Suppose that f is an odd degree polynomial with a positive leading coefficient. Then $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Then there exist sufficiently a large negative value of x (denoted by a) so that $f(a)$ is negative. Also, there exist sufficiently a large positive value of x (denoted by b) so that $f(b)$ is positive. Since f is a polynomial, it is continuous on $\mathbb{R}$ and thus on $[a, b]$. By the Intermediate Value Theorem, f must have a zero in the interval $[a, b]$. The proof goes similarly for functions with negative leading coefficients.