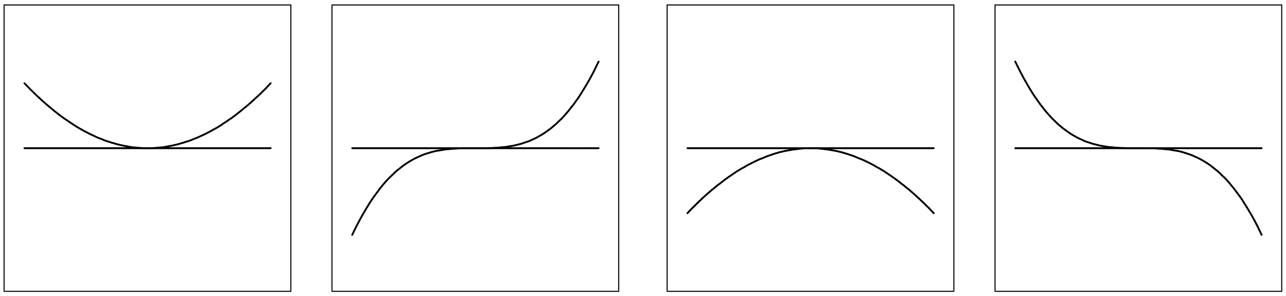
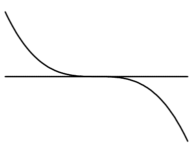
Now that we are usiing the chain rule, it became more complicated to sort out how the derivative changes sign around its zero. For such cases, we use the second derivative test.

Consider a function f that is twice differentiable on an open interval containing c. Suppose further that $f'(c) = 0$. Let us look at the function f' first. Since f is twice differentiable, f' is differentiable. Differentiable functions have just a few ways in which they take a zero value.



$f'(x)$ $f'(x)$ $f'(x)$ $f'(x)$

Case 1. Suppose that $f'(c) = 0$ and $f''(c)$ is negative.

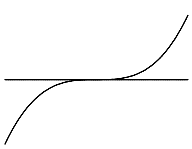


$f'(x)$

That indicates that f' is strictly decreasing on an interval containing c. Taking a zero value while decreasing means that f' changes sign from positive to negative. That indicates that f has a relative maximum at c.

Theorem: If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at c.

Case 2. Suppose that $f'(c) = 0$ and $f''(c)$ is positive.



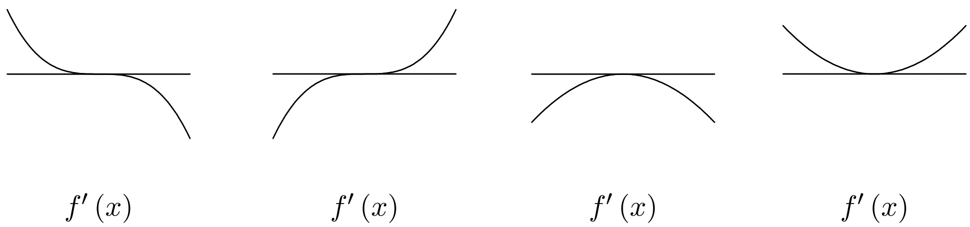
$f'(x)$

That indicates that f' is strictly increasing on an interval containing c. Taking a zero value while increasing means that f' changes sign from negative to positive. That indicates that f has a relative minimum at c.

Theorem: If $f'(c) = 0$ and $f''(c) > 0$, then f has a relative minimum at c.

Case 3. Suppose that $f'(c) = 0$ and $f''(c) = 0$. Consider $f(x) = x^8$ and $g(x) = x^9$ near $x = 0$.
f has a relative minimum at $x = 0$ and g has neither a maximum nor a minimum at $x = 0$, yet

$f'(0) = 0$, $f''(0) = 0$ and $g'(0) = 0$, $g''(0) = 0$



Furthermore, the higher order derivatives of f and g will be zero for quite a while. Therefore, the second derivative in this case did not distinguish between maximums and minimums.

Theorem: If $f'(c) = 0$ and $f''(c) = 0$, then the second derivative test did not yield for any useful result.

In such cases, we need to apply other methods to tell maximums, minimums, (or neither) apart.

Example 1. Suppose that $f(x) = \sin\left(\frac{1}{x}\right)$. Prove that f has a relative maximum at $x = \frac{2}{\pi}$.

Solution: We differentiate f and then f': $f'(x) = -\frac{1}{x^2}\cos\frac{1}{x}$ and $f''(x) = \frac{2}{x^3}\cos\frac{1}{x} - \frac{1}{x^4}\sin\frac{1}{x}$

Now we compute $f'\left(\frac{2}{\pi}\right)$ and $f'\left(\frac{2}{\pi}\right)$

$$f'\left(\frac{2}{\pi}\right) = -\frac{1}{\left(\frac{2}{\pi}\right)^2}\cos\left(\frac{1}{\left(\frac{2}{\pi}\right)}\right) = -\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}\right) = 0$$

and $f''\left(\frac{2}{\pi}\right) = 2\left(\frac{\pi}{2}\right)^3\cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)^4\sin\left(\frac{\pi}{2}\right) = -\frac{1}{16}\pi^4$

So we have that $f'\left(\frac{2}{\pi}\right) = 0$ and $f''\left(\frac{2}{\pi}\right)$ is negative. Therefore, by the second derivative test, f has a relative maximum at $\frac{2}{\pi}$.

The second derivative test worked in this case. Furthermore, we needed it, because it is difficult to sort out how $f'(x) = -\frac{1}{x^2}\cos\frac{1}{x}$ changes sign at $x = \frac{2}{\pi}$.

For more documents like this, visit our page at https://teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.