

The real numbers has the **completeness property**: If a set of real numbers is bounded above, it has a least upper bound; if a set of real numbers is bounded below, it has a greatest lower bound. (This is an axiom of the real numbers, and this is the one that distinguishes the set of rational numbers from the set of real numbers.)

Definition: A **sequence** is a list of numbers $a_1, a_2, a_3, \dots, a_n, \dots$ in a given order. The numbers a_n are **terms** of the sequence. The integer k is called the index of the term a_k .

An infinite sequence is a function with domain $\mathbb{N}$. We may start labeling at a number greater than 1.

We can describe sequences by writing rules

$$a_n = \sqrt{n} \quad b_n = (-1)^{n+1} \frac{1}{n} \quad c_n = \frac{n-1}{n} \quad d_n = (-1)^{n+1}$$

or listing the first few terms

$$\begin{aligned} \{a_n\} &= \{1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\} \\ \{b_n\} &= \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1} \frac{1}{n}, \dots\right\} \\ \{c_n\} &= \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n-1}{n}, \dots\right\} \\ \{d_n\} &= \{1, -1, 1, -1, \dots, (-1)^{n+1}, \dots\} \end{aligned}$$

Definition: The sequence $\{a_n\}$ **converges** to the number L if for every positive number ϵ there exists an integer N such that for all n ,

$$\text{if } n > N \text{ then } |a_n - L| < \epsilon.$$

If no such number L exists, we say $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L , we write $\lim_{n \rightarrow \infty} a_n = L$ or $a_n \rightarrow L$ and call L the **limit** of the sequence.

Example 1. a) $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

proof: Let $\epsilon > 0$ be given. Define $N = \left\lceil \frac{1}{\epsilon} \right\rceil + 1$.

If $n > N$, then $n > N > \frac{1}{\epsilon}$ and so $\frac{1}{n} < \epsilon$.

b) the constant sequence $\{a_n\} = \{c, c, c, \dots\}$. Clearly $a_n \rightarrow c$.

proof: Let $\epsilon > 0$ be given. Then $N = 1$ will do, because for all $n > 1$ we will have that $|c - c| = 0 < \epsilon$.

Example 2. $\{1, -1, 1, -1, 1, -1, \dots\}$ diverges.

proof: Suppose for a contradiction that such a number L exists. Let $\epsilon = \frac{1}{3}$.

There exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - L| < \frac{1}{3}$. Since 1 occurs in the sequence at arbitrarily high index, it must be that

$$\begin{aligned} |1 - L| &< \frac{1}{3} \\ -\frac{1}{3} &< L - 1 < \frac{1}{3} \\ \frac{2}{3} &< L < \frac{4}{3} \end{aligned}$$

Since -1 occurs in the sequence at arbitrarily high index, it also must be that

$$\begin{aligned} |-1 - L| &< \frac{1}{3} \\ -\frac{1}{3} &< L + 1 < \frac{1}{3} \\ -\frac{4}{3} &< L < -\frac{2}{3} \end{aligned}$$

There is no number L with $\frac{2}{3} < L < \frac{4}{3}$ and $-\frac{4}{3} < L < -\frac{2}{3}$, so the sequence diverges.

Example 3. The sequence $\{\sqrt{n}\}$ diverges differently.

Definition: The sequence $\{a_n\}$ **diverges to infinity** if for every real number M there exists an integer N such that for all n ,

$$\text{if } n > N \text{ then } a_n > M.$$

We denote this as $\lim_{n \rightarrow \infty} a_n = \infty$ or $a_n \rightarrow \infty$.

Similarly, the sequence $\{a_n\}$ **diverges to negative infinity** if for every real number m there exists an integer N such that for all n ,

$$\text{if } n > N \text{ then } a_n < m.$$

We denote this as $\lim_{n \rightarrow \infty} a_n = -\infty$ or $a_n \rightarrow -\infty$.

The sequence $\{\sqrt{n}\}$ diverges to infinity. The sequence $\{1, 0, 2, 0, 3, 0, \dots\}$ diverges, but does not diverge to infinity or negative infinity.

Theorem 1: Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Suppose that A and B are real numbers such that $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$. Then

1. Sum Rule: $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$
2. Difference Rule: $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$
3. Constant Multiple Rule: $\lim_{n \rightarrow \infty} (ca_n) = cA$ for all $c \in \mathbb{R}$
4. Product Rule: $\lim_{n \rightarrow \infty} (a_n b_n) = AB$
5. Quotient Rule: $\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \frac{A}{B}$ if $B \neq 0$

Example 4.

$$\text{a) } \lim_{n \rightarrow \infty} \frac{2}{n} = \lim_{n \rightarrow \infty} 2 \cdot \frac{1}{n} = 2 \lim_{n \rightarrow \infty} \frac{1}{n} = 2 \cdot 0 = 0$$

$$\text{b) } \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = \lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n} = 1 + 0 = 1$$

$$\text{c) } \lim_{n \rightarrow \infty} \left(-\frac{3}{n^2} \right) = -3 \lim_{n \rightarrow \infty} \left(\frac{1}{n} \cdot \frac{1}{n} \right) = -3 \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = -3 \cdot 0 \cdot 0 = 0$$

$$d) \lim_{n \rightarrow \infty} \frac{3 - 2n^4}{7n^4 + 2} = \lim_{n \rightarrow \infty} \frac{\frac{3}{n^4} - 2}{7 + \frac{2}{n^4}} = \frac{\lim_{n \rightarrow \infty} \left(\frac{3}{n^4} - 2 \right)}{\lim_{n \rightarrow \infty} \left(7 + \frac{2}{n^4} \right)} = \frac{-2}{7} = -\frac{2}{7}$$

Theorem 2. (The Sandwich Theorem for Sequences) Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences with $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$. Suppose that there exists N positive integer such that for all $n > N$,

$$a_n \leq b_n \leq c_n$$

then b_n converges to L .

Consequence: If $|b_n| \leq c_n$ and $c_n \rightarrow 0$, then $b_n \rightarrow 0$.

Example 5.

$$a) \frac{\sin n}{n} \rightarrow 0 \text{ since } -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \text{ and } \frac{1}{n} \rightarrow 0 \text{ and } -\frac{1}{n} \rightarrow 0$$

$$b) \frac{(-1)^n}{n^2} \rightarrow 0 \text{ since } -\frac{1}{n^2} \leq \frac{(-1)^n}{n^2} \leq \frac{1}{n^2}$$

Theorem 3 (The Continuous Function Theorem for Sequences). Let $\{a_n\}$ be a sequence of real numbers.

If $a_n \rightarrow L$ and if f is a function that is continuous at L and defined at all a_n , then $f(a_n) \rightarrow f(L)$.

Example 6. a) $\sqrt{\frac{n+1}{n}} \rightarrow 1$

proof: $\frac{n+1}{n} \rightarrow 1$ and $f(x) = \sqrt{x}$ is continuous at $x = 1$. Thus $\sqrt{\frac{n+1}{n}} \rightarrow \sqrt{1} = 1$

$$b) \ln \left(\frac{n^2 - 1}{n^2 + 1} \right) \rightarrow \ln 1 = 0$$

$$c) 2^{1/n} \rightarrow 2^0 = 1$$

Theorem 4: Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\text{if } \lim_{x \rightarrow \infty} f(x) = L, \text{ then } \lim_{n \rightarrow \infty} a_n = L$$

Then we can apply L'Hôpital's rule to find limits of sequences.

Example 7. a) $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$

proof: $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0$

$$b) \lim_{n \rightarrow \infty} \left(\frac{n+1}{n-1} \right)^n = e^2$$

proof: $\ln a_n = \ln \left(\frac{n+1}{n-1} \right)^n = n \ln \left(\frac{n+1}{n-1} \right) = \frac{\ln \left(\frac{n+1}{n-1} \right)}{\frac{1}{n}}$

This is an indeterminate of the type $\frac{0}{0}$

$$\lim_{n \rightarrow \infty} \ln a_n = \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{n+1}{n-1} \right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{-\frac{2}{n^2-1}}{-\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \left(-\frac{2}{n^2-1} \right) (-n^2) = \lim_{n \rightarrow \infty} \frac{2n^2}{n^2-1} = 2$$

$\ln a_n \rightarrow 2$ and $f(x) = e^x$ is continuous on $\mathbb{R}$. thus $a_n = e^{\ln a_n} \rightarrow e^2$

Commonly Occurring Limits. In each of the following, c is a fixed number.

1. $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$
2. $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$
3. $\lim_{n \rightarrow \infty} x^{1/n} = 1 \quad x > 0$
4. $\lim_{n \rightarrow \infty} x^n = 0 \quad -1 < x < 1$
5. $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n} \right)^n = e^x \quad (\text{any } x)$
6. $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

proof. 1) was done using L'Hôpital's rule

$$2) \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

proof: $\lim_{n \rightarrow \infty} \sqrt[n]{n} = \lim_{n \rightarrow \infty} n^{1/n} = \lim_{x \rightarrow \infty} x^{1/x}$

$$\lim_{x \rightarrow \infty} \ln(x^{1/x}) = \lim_{x \rightarrow \infty} \frac{1}{x} \ln x = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0$$

$f(x) = e^x$ is continuous on $\mathbb{R}$

$\ln(x^{1/x}) \rightarrow 0$ and so $e^{\ln(x^{1/x})} \rightarrow e^0$ and so $x^{1/x} \rightarrow 1$