

Continuity

Definition: (*Interior point*) A function $y = f(x)$ is continuous at an interior point c of its domain if $\lim_{x \rightarrow c} f(x) = f(c)$.

(*Endpoint*) A function $y = f(x)$ is continuous at a left endpoint a or is continuous at a right endpoint b of its domain if

$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \rightarrow b^-} f(x) = f(b), \quad \text{respectively.}$$

Theorem: Properties of continuous functions?

Theorem: If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c .

Theorem: If g is continuous at the point b and $\lim_{x \rightarrow c} f(x) = b$, then

$$\lim_{x \rightarrow c} g(f(x)) = g(b) = g\left(\lim_{x \rightarrow c} f(x)\right)$$

Theorem: (**The Intermediate value Theorem for Continuous Functions**) If f is continuous on a closed interval $[a, b]$ and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

Maximum-minimum theorems

Theorem: (**Extreme Value Theorem**) If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Definition: A critical number of a function f is a number c in its domain such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem: (Fermat) If f has a local maximum or minimum at c , then c is a critical number of f .

Closed interval method: To find absolute extrema of a continuous function f on a closed interval $[a, b]$.

- 1) Find the values of f at the critical numbers of f in $[a, b]$.
- 2) Find the values of f at the endpoints of the interval.
- 3) The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

The Mean Value Theorem

Theorem: (Rolle) Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) . If $f(a) = f(b)$, then there exists c in (a, b) with $f'(c) = 0$.

Theorem: (Mean Value Theorem) Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) . Then there exists c in (a, b) with $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Corollary: If $f'(x) = 0$ on an interval (a, b) , then $f(x) = c$ on (a, b) for a constant c .

Theorem: If $f'(x) = g'(x)$ on (a, b) , then there exists c with $f(x) = g(x) + c$ on (a, b) .

The Fundamental Theorem

Theorem: (**Mean value Theorem for Definite Integrals**) If f is continuous on $[a, b]$, then there exists c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

Theorem: (**Fundamental Theorem of Calculus, Part 1**) If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$, differentiable on (a, b) , and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Theorem: (**Fundamental Theorem of Calculus, Part 2**) If f is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Theorem: (Net Change Theorem) The net change in a function $F(x)$ over an interval $[a, b]$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x) dx$$