

Theorem: If f is differentiable at a , then it is continuous there.

Proof: Suppose that f is differentiable at a number a . Then $f'(a)$ exists which means that $f(a)$ exists and the limit $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ also exists and is finite. Let us start with the true statement that $0 = 0 \cdot f'(a)$.

$$\begin{aligned}
 0 &= 0 \cdot f'(a) \\
 0 &= \lim_{h \rightarrow 0} h \cdot \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} && \text{by the product rule of limits} \\
 0 &= \lim_{h \rightarrow 0} \left(h \cdot \frac{f(a+h) - f(a)}{h} \right) && \text{cancel out } h \\
 0 &= \lim_{h \rightarrow 0} (f(a+h) - f(a)) && \text{by the difference rule of limits} \\
 0 &= \lim_{h \rightarrow 0} f(a+h) - \lim_{h \rightarrow 0} f(a) \\
 \lim_{h \rightarrow 0} f(a) &= \lim_{h \rightarrow 0} f(a+h) && \text{by the constant rule of limits} \\
 f(a) &= \lim_{h \rightarrow 0} f(a+h)
 \end{aligned}$$

and $f(a) = \lim_{h \rightarrow 0} f(a+h)$ means that f is continuous at a .

Theorem: (Product rule of derivatives) If f and g are differentiable at x , then so is fg and the derivative is

$$(fg)' = f'g + fg'$$

Proof: Suppose that f and g are differentiable at x .

$$\begin{aligned}
 (fg)' &= \lim_{h \rightarrow 0} \frac{fg(x+h) - fg(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} && \text{smuggle in } f(x+h)g(x) \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h} \\
 &= \lim_{h \rightarrow 0} f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} && \text{by the sum rule of limits} \\
 &= \lim_{h \rightarrow 0} f(x+h) \frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} g(x) \frac{f(x+h) - f(x)}{h} && \text{by the product rule of limits} \\
 &= \lim_{h \rightarrow 0} f(x+h) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} g(x) \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{by the constant rule of limits} \\
 &= \lim_{h \rightarrow 0} f(x+h) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}
 \end{aligned}$$

We now realize the following:

$$\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = g'(x) \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Furthermore, f is continuous because it is differentiable, and so $\lim_{h \rightarrow 0} f(x+h) = f(x)$. Thus we now have that

$$\begin{aligned}
 (fg)' &= \lim_{h \rightarrow 0} f(x+h) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= f(x) \cdot g'(x) + g(x) \cdot f'(x)
 \end{aligned}$$