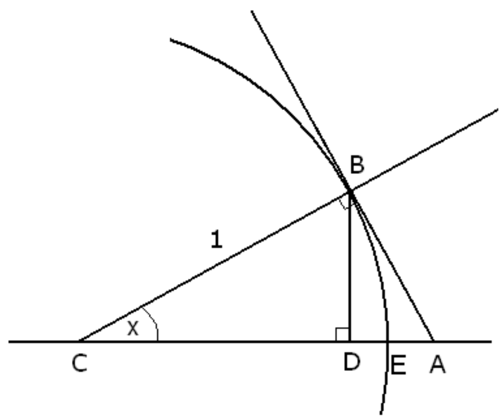
Theorem 1: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Proof: This theorem and the next one are necessary for differentiating $\sin x$ and $\cos x$. Recall a theorem: Let r be the radius of a circle. If α is measured in radians, then the area of a sector with a central angle of α is $A_{\text{sector}} = \frac{\alpha r^2}{2}$. (Notation: $\overline{AB}$ will denote the length of line segment AB.)

Let x be a very small positive angle, measured in radians, drawn into a unit circle as shown on the picture below. Let B be the point where the unit circle intersects the ray determined by x. We then draw a tangent line to the circle at point B. Let A be the point where the tangent line intersects the $x-$axis. We also draw a vertical line through B. Let D be the point where this vertical line intersects the $x-$axis. Finally, let us denote by E the point with coordinates $(0, 1)$.



The proof will be based on the following fact: because they include each other, the following three areas can be easily compared:

$$\text{Area of triangle } CDB \ \le\ \text{Area of sector } CEB \ \le\ \text{Area of triangle } ABC$$

Area of triangle CDB: the horizontal side, $\overline{CD} = \cos x$ and the vertical side, $\overline{DB} = \sin x$. Since this is a right triangle, the area is: $A_{CDB} = \frac{1}{2}\sin x \cos x$

Area of sector CEB: $A_{\text{sector}} = \frac{1^2 x}{2} = \frac{x}{2}$

Area of triangle ABC: there is a right angle at point B because the tangent line drawn to a circle is perpendicular to the radius drawn to the point of tangency. So the area is $A_{ABC} = \frac{1}{2}\overline{AB}\cdot\overline{BC}$. Clearly $\overline{BC} = 1$. To compute $\overline{AB}$, in triangle ABC, $\tan x = \frac{\overline{AB}}{1}$ and so $\overline{AB} = \tan x$.

Area of triangle ABC: $\frac{1}{2}(1)(\tan x) = \frac{\tan x}{2}$ or $\frac{\sin x}{2\cos x}$. So now

$$\text{Area of triangle } CDB \ \le\ \text{Area of sector } CEB \ \le\ \text{Area of triangle } ABC$$

translates to

$$\frac{1}{2}\sin x \cos x \le \frac{x}{2} \le \frac{\sin x}{2\cos x}$$

Let us divide all three sides by $\frac{\sin x}{2}$. Because x is small and positive, $\frac{\sin x}{2}$ is positive and so we do not need to reverse the inequality signs.

$$\cos x \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

Suppose now that x approaches zero. Then both $\cos x$ and $\dfrac{1}{\cos x}$ approach 1. By the sandwich principle, $\dfrac{x}{\sin x}$, the quantity locked in between those two must also approach 1.

$$\begin{array}{ccccc} \cos x & \leq & \dfrac{x}{\sin x} & \leq & \dfrac{1}{\cos x} \\ \downarrow & & & & \downarrow \\ 1 & & & & 1 \end{array}$$

If $\dfrac{x}{\sin x}$ approaches 1, so does its reciprocal, $\dfrac{\sin x}{x}$.

So far, we have proven the statement for positive values of x, that is, $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{x} = 1$. A similar argument works for negative values of x.

Theorem 2: $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

Proof:

$$\begin{aligned} \lim_{x\to 0} \frac{\cos x - 1}{x} &= \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot 1 = \lim_{x\to 0} \left(\frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} \right) = \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x\to 0} \frac{-\left(1 - \cos^2 x\right)}{x(\cos x + 1)} \\ &= \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x + 1)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{-\sin x}{\cos x + 1} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{-\sin x}{\cos x + 1} = 1 \cdot 0 = 0 \end{aligned}$$

For more documents like this, visit our page at http://www.teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.

