

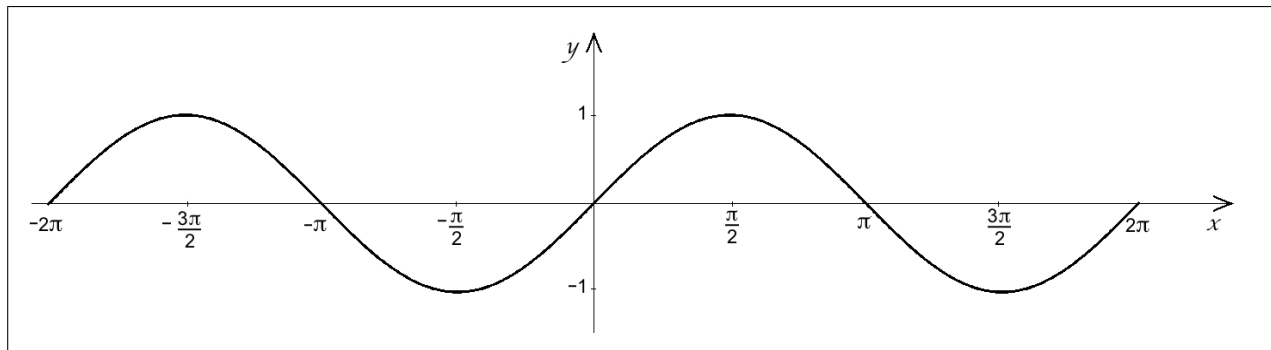
$$f(x) = \sin x$$

domain: $\mathbb{R}$

range: $[-1, 1]$

periodic with period 2π : for all x , $\sin(x + 2\pi) = \sin x$

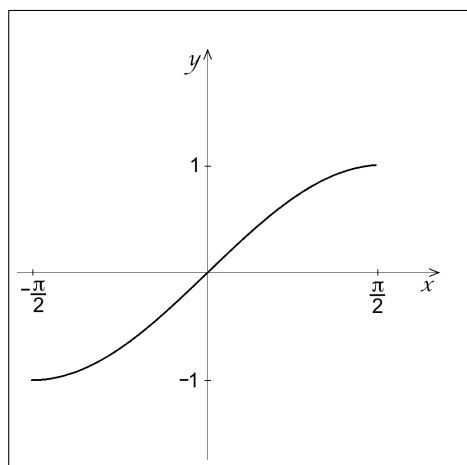
odd function: for all x , $\sin(-x) = -\sin x$



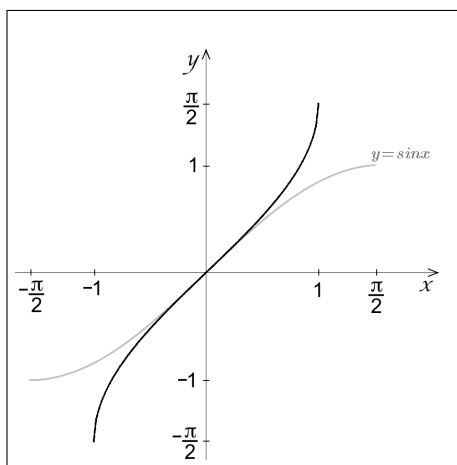
inverse function: $y = \sin^{-1} x = \arcsin x$

domain: $[-1, 1]$

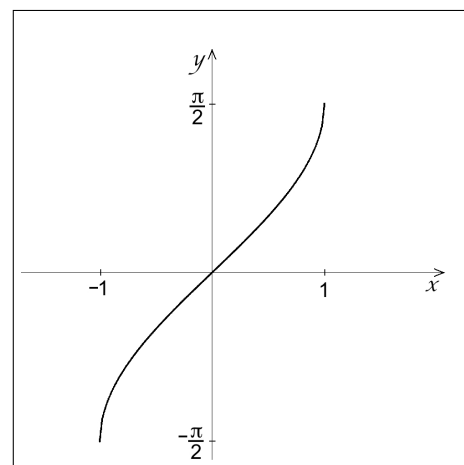
range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Restrict the domain of $\sin x$
to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Reverse function assignment

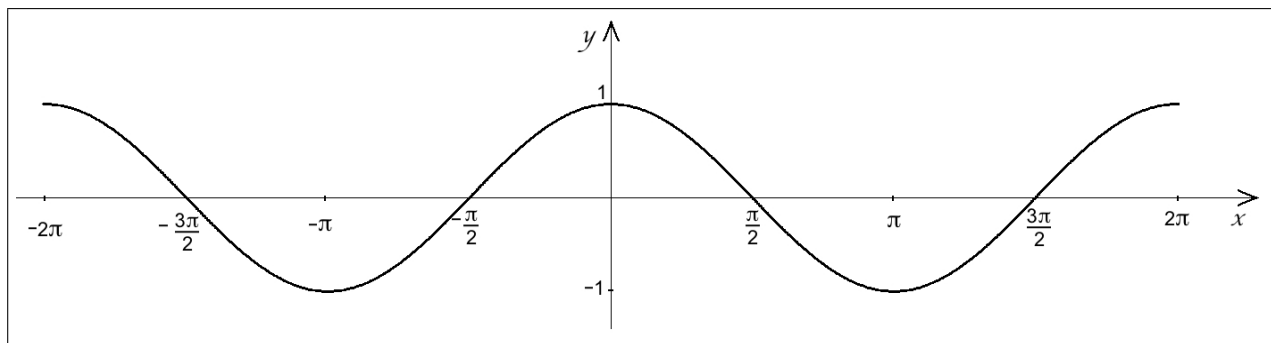


The graph of $y = \sin^{-1} x$

$$f(x) = \cos x$$

domain: $\mathbb{R}$
range: $[-1, 1]$

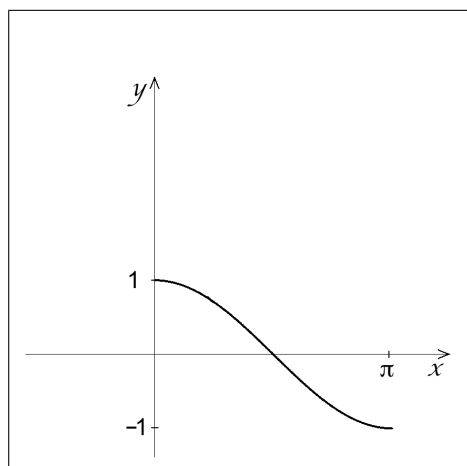
periodic with period 2π : for all x , $\cos(x + 2\pi) = \cos x$
even function: for all x , $\cos(-x) = \cos x$



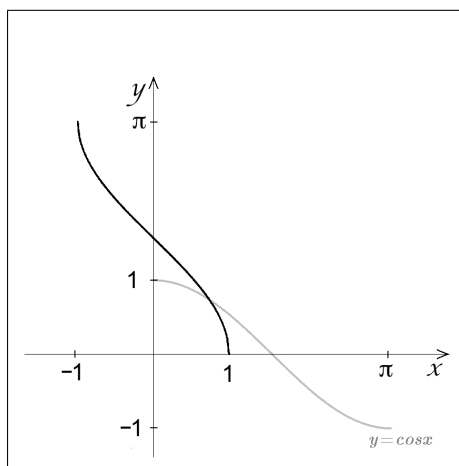
inverse function: $y = \cos^{-1} x = \arccos x$

domain: $[-1, 1]$

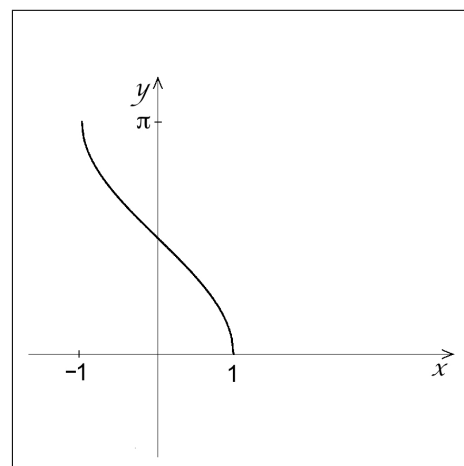
range: $[0, \pi]$



Restrict the domain of $\cos x$
to $[0, \pi]$



Reverse function assignment



The graph of $y = \cos^{-1} x$

Theorem:

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2} \quad \text{and} \quad \cos^{-1} x + \cos^{-1}(-x) = \pi$$

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

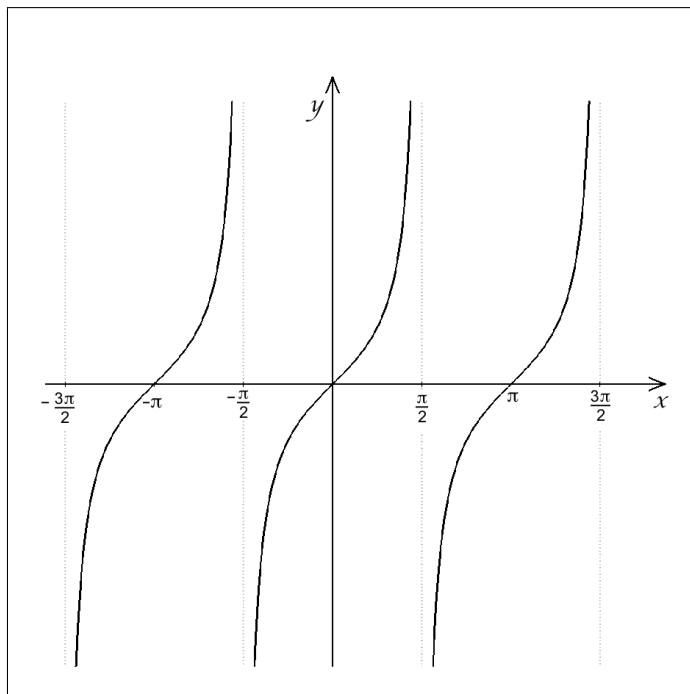
domain: $x \neq \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

vertical asymptotes at $x = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

range: $\mathbb{R}$

periodic with period π : for all x , $\tan(x + \pi) = \tan x$

odd function: for all x , $\tan(-x) = -\tan x$



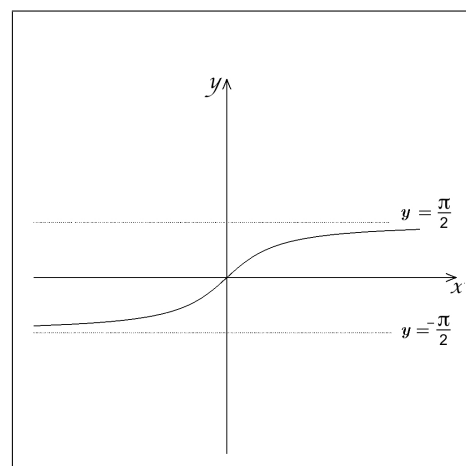
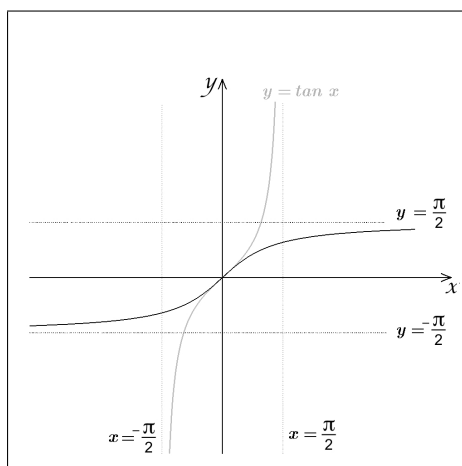
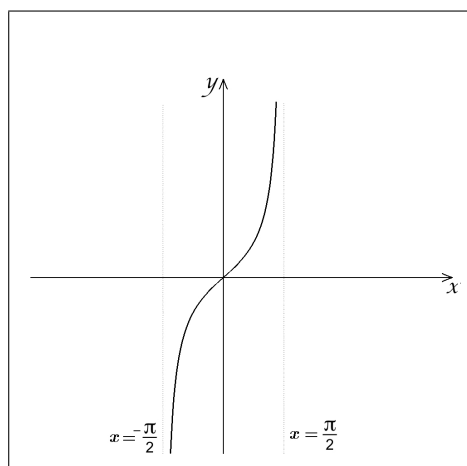
inverse function: $y = \tan^{-1} x = \arctan x$

domain: $\mathbb{R}$

$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2} \quad \text{and} \quad \lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$$

range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

horizontal asymptotes: $y = \pm \frac{\pi}{2}$



Restrict the domain of $\tan x$
to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Reverse function assignment

The graph of $y = \tan^{-1} x$

$$f(x) = \csc x = \frac{1}{\sin x}$$

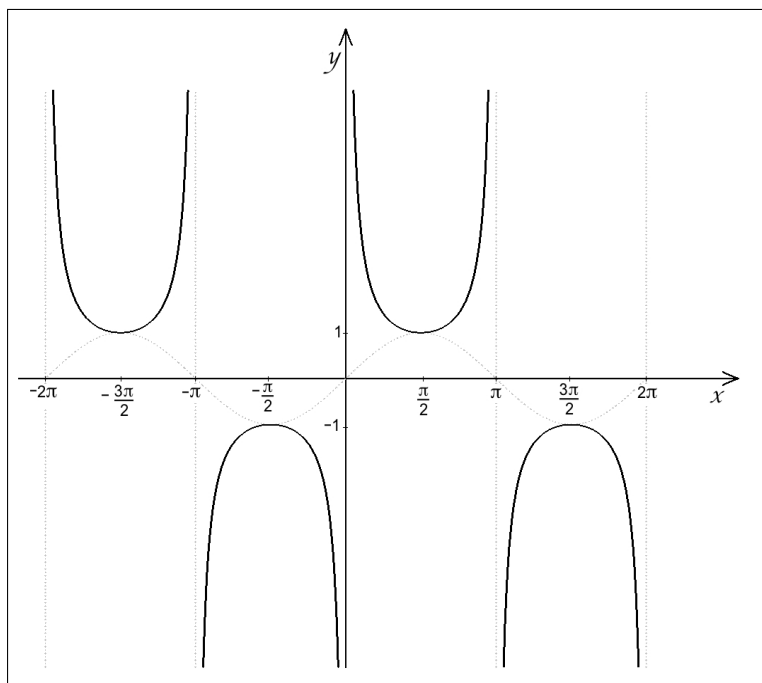
domain: $x \neq k\pi$

vertical asymptotes at $x = k\pi$ where $k \in \mathbb{Z}$

range: $(-\infty, -1] \cup [1, \infty)$

periodic with period 2π : for all x , $\csc(x + 2\pi) = \csc x$

odd function: for all x , $\csc(-x) = -\csc x$



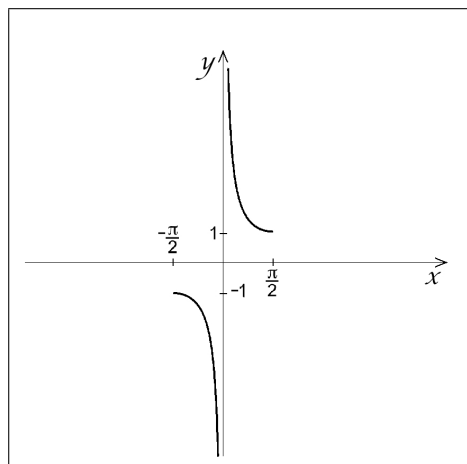
inverse function: $y = \csc^{-1} x = \operatorname{arccsc} x$

domain: $(-\infty, -1] \cup [1, \infty)$

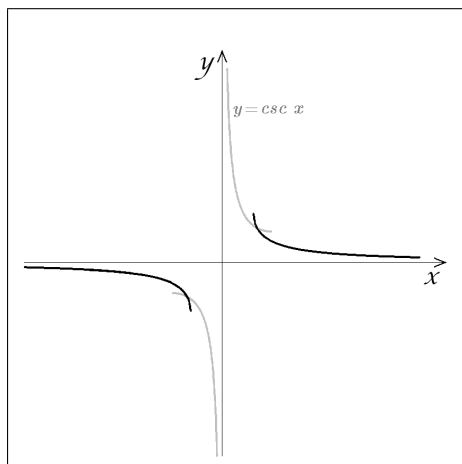
$$\lim_{x \rightarrow -\infty} \csc^{-1} x = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \csc^{-1} x = 0$$

range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

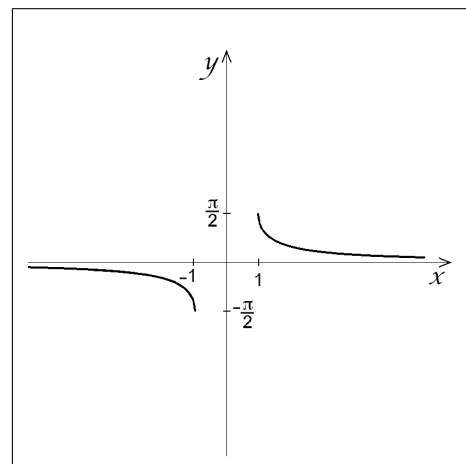
horizontal asymptote: $y = 0$



Restrict the domain of $\csc x$ to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



Reverse function assignment



The graph of $y = \csc^{-1} x$

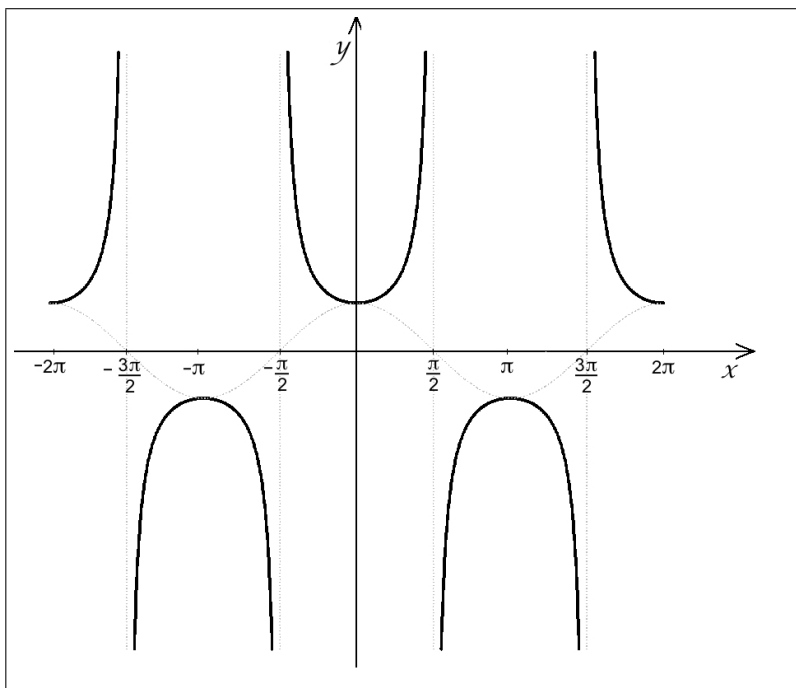
$$f(x) = \sec x = \frac{1}{\cos x}$$

domain: $x \neq \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

vertical asymptotes at $x = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

range: $(-\infty, -1] \cup [1, \infty)$

periodic with period 2π : for all x , $\sec(x + 2\pi) = \sec x$ even function: for all x , $\sec(-x) = \sec x$



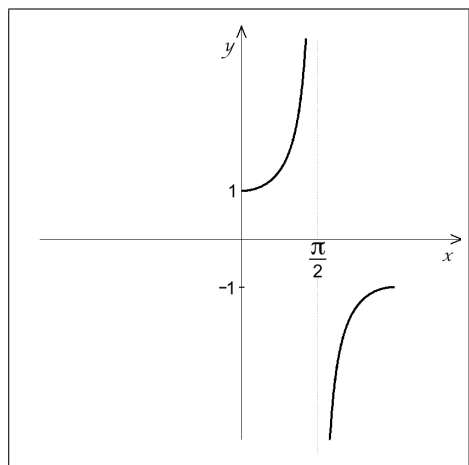
inverse function: $y = \sec^{-1} x = \operatorname{arcsec} x$

domain: $(-\infty, -1] \cup [1, \infty)$

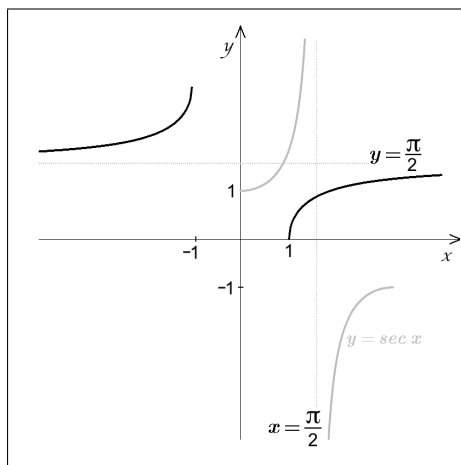
$$\lim_{x \rightarrow -\infty} \sec^{-1} x = \frac{\pi}{2} \quad \text{and} \quad \lim_{x \rightarrow \infty} \sec^{-1} x = \frac{\pi}{2}$$

range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

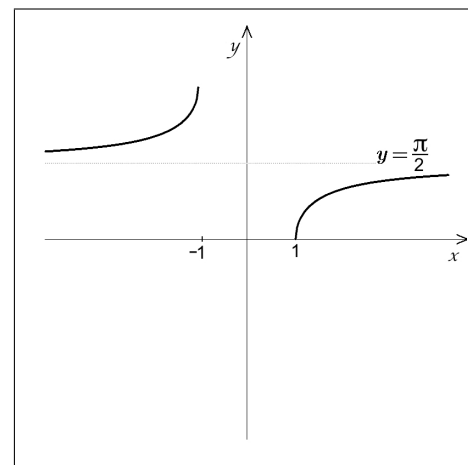
horizontal asymptote: $y = \frac{\pi}{2}$



Restrict the domain of $\sec x$
to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



Reverse function assignment



The graph of $y = \sec^{-1} x$

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

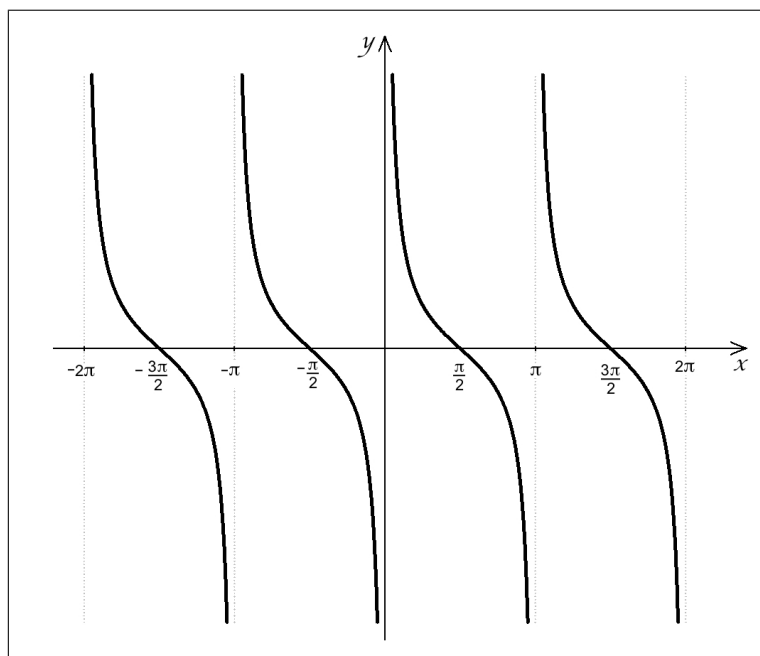
domain: $x \neq k\pi$

vertical asymptotes at $x = k\pi$ where $k \in \mathbb{Z}$

range: $\mathbb{R}$

periodic with period π : for all x , $\cot(x + \pi) = \cot x$

odd function: for all x , $\cot(-x) = -\cot x$



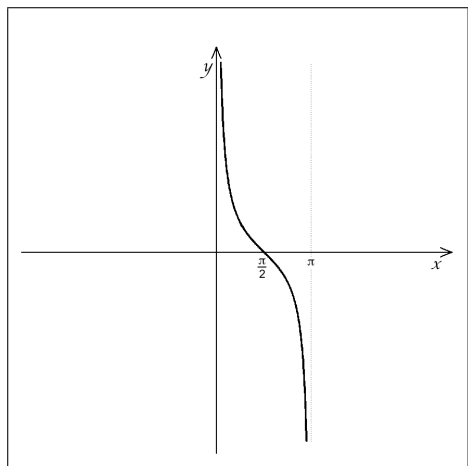
inverse function: $y = \cot^{-1} x = \operatorname{arccot} x$

domain: $\mathbb{R}$

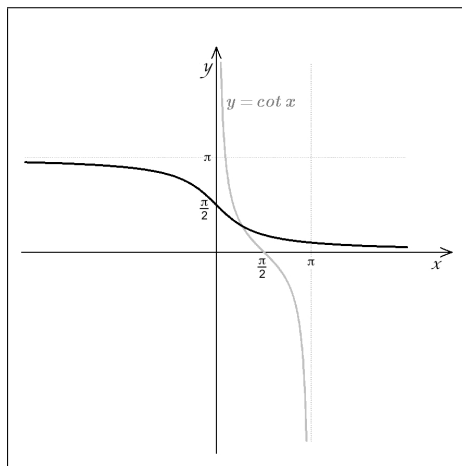
$$\lim_{x \rightarrow -\infty} \cot^{-1} x = \pi \quad \text{and} \quad \lim_{x \rightarrow \infty} \cot^{-1} x = 0$$

range: $(0, \pi)$

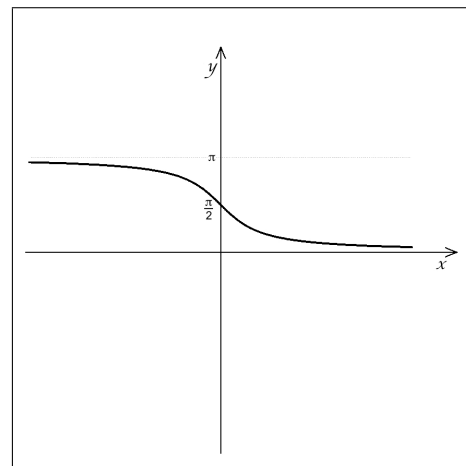
horizontal asymptote: $y = 0$ and $y = \pi$



Restrict the domain of $\cot x$ to $(0, \pi)$.



Reverse function assignment



The graph of $y = \cot^{-1} x$

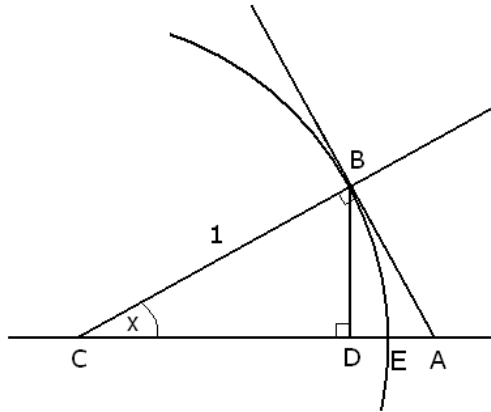
Proofs

Theorems 1 and 2: $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$

Claim 1.) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Proof: This theorem and the next one are necessary for differentiating $\sin x$ and $\cos x$. Recall a theorem: Let r be the radius of a circle. If α is measured in radians, then the area of a sector with a central angle of α is $A_{\text{sector}} = \frac{\alpha r^2}{2}$. (Notation: $\overline{AB}$ will denote the length of line segment AB .)

Let x be a very small positive angle, measured in radians, drawn into a unit circle as shown on the picture below. Let B be the point where the unit circle intersects the ray determined by x . We then draw a tangent line to the circle at point B . Let A be the point where the tangent line intersects the x -axis. We also draw a vertical line through B . Let D be the point where this vertical line intersects the x -axis. Finally, let us denote by E the point with coordinates $(0, 1)$.



The proof will be based on the following fact: because they include each other, the following three areas can be easily compared:

$$\text{Area of triangle } CDB \leq \text{Area of sector } CEB \leq \text{Area of triangle } ABC$$

Area of triangle CDB : the horizontal side, $\overline{CD} = \cos x$ and the vertical side, $\overline{DB} = \sin x$. Since this is a right triangle, the area is: $A_{CDB} = \frac{1}{2} \sin x \cos x$.

Area of sector CEB : $A_{\text{sector}} = \frac{1^2 x}{2} = \frac{x}{2}$.

Area of triangle ABC : there is a right angle at point B because the tangent line drawn to a circle is perpendicular to the radius drawn to the point of tangency. So the area is $A_{ABC} = \frac{1}{2} \overline{AB} \cdot \overline{BC}$. Clearly $\overline{BC} = 1$. To compute

$\overline{AB}$, in triangle ABC , $\tan x = \frac{\overline{AB}}{1}$ and so $\overline{AB} = \tan x$.

Area of triangle ABC : $\frac{1}{2} (1) (\tan x) = \frac{\tan x}{2}$ or $\frac{\sin x}{2 \cos x}$. So now

$$\text{Area of triangle } CDB \leq \text{Area of sector } CEB \leq \text{Area of triangle } ABC$$

translates to

$$\frac{1}{2} \sin x \cos x \leq \frac{x}{2} \leq \frac{\sin x}{2 \cos x}$$

Let us divide all three sides by $\frac{\sin x}{2}$. Because x is small and positive, $\frac{\sin x}{2}$ is positive and so we do not need to reverse the inequality signs.

$$\cos x \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

Suppose now that x approaches zero. Then both $\cos x$ and $\frac{1}{\cos x}$ approach 1. By the sandwich principle, $\frac{x}{\sin x}$, the quantity locked in between those two must also approach 1.

$$\begin{array}{ccccc} \cos x & \leq & \frac{x}{\sin x} & \leq & \frac{1}{\cos x} \\ \downarrow & & & & \downarrow \\ 1 & & & & 1 \end{array}$$

If $\frac{x}{\sin x}$ approaches 1, so is its reciprocal, $\frac{\sin x}{x}$.

So far, we have proven the statement for positive values of x , that is, $\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$. A similar argument works for negative values of x .

Claim 2.) $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$

Proof:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} &= \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} \cdot 1 = \lim_{x \rightarrow 0} \left(\frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} \right) = \lim_{x \rightarrow 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x \rightarrow 0} \frac{-(1 - \cos^2 x)}{x(\cos x + 1)} \\ &= \lim_{x \rightarrow 0} \frac{-\sin^2 x}{x(\cos x + 1)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{-\sin x}{\cos x + 1} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{-\sin x}{\cos x + 1} = 1 \cdot 0 = 0 \end{aligned}$$

We are now ready to prove that $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$

Proof:

$$\begin{aligned} \frac{d}{dx}(\sin x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) = \lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1)}{h} + \lim_{h \rightarrow 0} \cos x \frac{\sin h}{h} \\ &= \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}(\cos x) &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \right) = \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1)}{h} - \lim_{h \rightarrow 0} \frac{\sin x \sin h}{h} \\ &= \cos x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} = \cos x \cdot 0 - \sin x \cdot 1 = -\sin x \end{aligned}$$

Theorem 3 and 4: $\frac{d}{dx}(\tan x) = \sec^2 x = \tan^2 x + 1$ and $\frac{d}{dx}(\cot x) = -\csc^2 x = -\cot^2 x - 1$.

Proof: We write $\tan x = \frac{\sin x}{\cos x}$ and apply the quotient rule.

$$\frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = \frac{\left(\frac{d}{dx} \sin x \right) \cos x - \left(\frac{d}{dx} \cos x \right) \sin x}{\cos^2 x} = \frac{\cos x \cos x - (-\sin x) \sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

We will now prove $\frac{1}{\cos^2 x} = \tan^2 x + 1$, which is a very important connection. Looking at the previous computation,

$$\frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

The proof for $\frac{d}{dx}(\cot x) = -\cot^2 x - 1 = -\csc^2 x$ is very similar. We apply the quotient rule.

$$\begin{aligned} \frac{d}{dx}(\cot x) &= \frac{d}{dx} \left(\frac{\cos x}{\sin x} \right) = \frac{\left(\frac{d}{dx} \cos x \right) \sin x - \cos x \left(\frac{d}{dx} \sin x \right)}{\sin^2 x} = \frac{-\sin x \sin x - \cos x (\cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} = -\csc^2 x \end{aligned}$$

Also,

$$\frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-\sin^2 x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} = -1 - \cot^2 x$$

Theorems 5 and 6: $\frac{d}{dx}(\sec x) = \sec x \tan x$ and $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Proof: We write $\sec x = \frac{1}{\cos x} = (\cos x)^{-1}$ and apply the chain rule.

$$\frac{d}{dx}(\sec x) = \frac{d}{dx} \left((\cos x)^{-1} \right) = -1 (\cos x)^{-2} \left(\frac{d}{dx}(\cos x) \right) = \frac{-1}{\cos^2 x} (-\sin x) = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

The proof for $\frac{d}{dx} \csc x$ is virtually identical: we apply the chain rule.

$$\frac{d}{dx}(\csc x) = \frac{d}{dx} \left((\sin x)^{-1} \right) = -1 (\sin x)^{-2} \left(\frac{d}{dx} \sin x \right) = \frac{-1}{\sin^2 x} \cos x = -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\cot x \csc x$$

Note: why do we prefer the form $\sec x \tan x$ over the form $\frac{\sin x}{\cos^2 x}$? One of the reasons is the advantage we'll see in differentiating the inverse functions $\sec^{-1} x$ and $\csc^{-1} x$.

Theorems 7 and 8: $\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
--

Proof: Recall that when we compose a function f with its inverse f^{-1} , the result is always the same function, (also called the identity function, $id(x) = x$)

$$f(f^{-1}(x)) = x$$

We will state this fact for $f(x) = \sin x$ and differentiate both sides of the equation. For the left-hand side, we use the chain rule.

$$\begin{aligned} \sin(\sin^{-1} x) &= x \\ \cos(\sin^{-1} x) \cdot \frac{d}{dx} \sin^{-1} x &= 1 && \text{divide by } \cos(\sin^{-1} x) \\ \frac{d}{dx} \sin^{-1} x &= \frac{1}{\cos(\sin^{-1} x)} \end{aligned}$$

We now need to simplify $\cos(\sin^{-1} x)$. We will present two methods to simplify this expression.

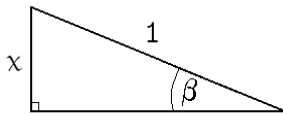
Method 1. We first introduce a new variable, β . Let $\beta = \sin^{-1} x$. This means that $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, and $\sin \beta = x$. We need to simplify $\underbrace{\cos(\sin^{-1} x)}_{\beta} = \cos \beta$. Since $\sin^2 \beta + \cos^2 \beta = 1$,

$$\cos \beta = \pm \sqrt{1 - \sin^2 \beta} = \pm \sqrt{1 - x^2}$$

Since $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, $\cos \beta$ is positive and so $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

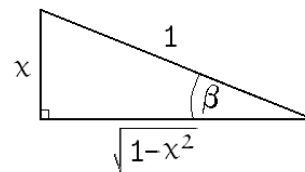
Method 2. We first introduce a new variable, β . Let $\beta = \sin^{-1} x$. This means that $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, and $\sin \beta = x$. Now the goal is to simplify $\underbrace{\cos(\sin^{-1} x)}_{\beta} = \cos \beta$.

We will use a right triangle to find the expression - up to its sign. We first draw a right triangle in which $\sin \beta = x = \frac{x}{1}$.



Next we use the Pythagorean Theorem to find the missing side to be $\sqrt{1 - x^2}$.

The advantage of this method is that now we can read *any* trigonometric function value of $\beta = \sin^{-1} x$ using this right triangle.



From the triangle,

$$\cos \beta = \cos(\sin^{-1} x) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

The answer at this point is really $\pm \sqrt{1 - x^2}$ as the triangle gave us the answer only up to a sign. For the sign, we need to argue using the location of β on the unit circle. Since $\beta = \sin^{-1} x$, $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$. Thus β is in the first or in the fourth quadrant. In both cases, cosine is positive, thus $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

Thus $\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}}.$

The proof for $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ is virtually identical.

Proof: Recall that when we compose a function f with its inverse f^{-1} , the result is always the same function.

$$f(f^{-1}(x)) = x$$

We will state this fact for $f(x) = \cos x$ and differentiate both sides of the equation. For the left-hand side, we use the chain rule.

$$\begin{aligned} \cos(\cos^{-1} x) &= x \\ -\sin(\cos^{-1} x) \cdot \frac{d}{dx}(\cos^{-1} x) &= 1 && \text{divide by } \sin(\cos^{-1} x) \\ \frac{d}{dx}(\cos^{-1} x) &= -\frac{1}{\sin(\cos^{-1} x)} \end{aligned}$$

We now need to simplify the expression $\sin(\cos^{-1} x)$. We will present two methods for this.

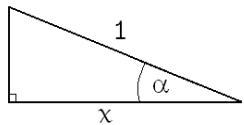
Method 1. Let $\alpha = \cos^{-1} x$. Then $x = \cos \alpha$ and α is between 0 and π .

$$\sin\left(\underbrace{\cos^{-1} x}_{\alpha}\right) = \sin \alpha = \pm \sqrt{1 - \cos^2 \alpha} = \pm \sqrt{1 - x^2}$$

Since α is between 0 and π , $\sin \alpha$ is positive and so $\sin \alpha = \sqrt{1 - x^2}$.

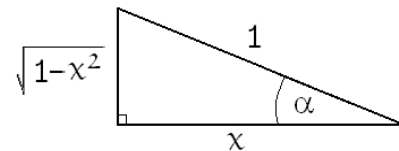
Method 2. Let $\alpha = \cos^{-1} x$. Then $x = \cos \alpha$ and α is between 0 and π .

We first draw a triangle in which $\cos \alpha = x = \frac{x}{1}$. Please note that every time we approach such a trigonometric question using a right triangle, our answer would be accurate up to sign - for the sign we would have to argue separately.



We find the missing side via the Pythagorean Theorem: $\sqrt{1 - x^2}$.

Now we can read *any* trigonometric function value using this triangle.



Now we read sine from the triangle:

$$\sin \alpha = \sin(\cos^{-1} x) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

The answer at this point is really $\pm\sqrt{1-x^2}$ as the triangle gave us the answer only up to a sign. For the sign, we need to argue using the location of α on the unit circle. Since $\alpha = \cos^{-1} x$, $0 \leq \alpha \leq \pi$. Thus α is in the first or in the second quadrant. In both cases, sine is positive, thus $\sin(\cos^{-1} x) = \sqrt{1-x^2}$.

Consequently, $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1-x^2}}$



Enrichment

If $\frac{d}{dx}(\sin^{-1} x)$ and $\frac{d}{dx}(\cos^{-1} x)$ are opposites, then what can be said about the function $f(x) = \sin^{-1} x + \cos^{-1} x$?

Theorems 9 and 10: $\frac{d}{dx} (\tan^{-1} x) = \frac{1}{x^2 + 1}$ and $\frac{d}{dx} (\cot^{-1} x) = -\frac{1}{x^2 + 1}$

Proof: Recall that $\frac{d}{dx} (\tan x) = \sec^2 x = \tan^2 x + 1$. Also recall that when we compose a function f with its inverse f^{-1} , the result is always the same function.

$$f(f^{-1}(x)) = x$$

We will state this fact for $f(x) = \tan x$ and differentiate both sides of the equation. For the left-hand side, we use the chain rule.

$$\begin{aligned} \tan(\tan^{-1} x) &= x \\ \sec^2(\tan^{-1} x) \cdot \frac{d}{dx}(\tan^{-1} x) &= 1 \\ (\tan^2(\tan^{-1} x) + 1) \cdot \frac{d}{dx}(\tan^{-1} x) &= 1 & \tan(\tan^{-1} x) = x \\ (x^2 + 1) \cdot \frac{d}{dx}(\tan^{-1} x) &= 1 & \text{divide by } x^2 + 1 \\ \frac{d}{dx}(\tan^{-1} x) &= \frac{1}{x^2 + 1} \end{aligned}$$

The proof for $\frac{d}{dx} (\cot^{-1} x) = -\frac{1}{x^2 + 1}$ is virtually identical. We compose the function $\cot x$ with its inverse $\cot^{-1} x$ and differentiate. Recall that $\frac{d}{dx} \cot x = -\cot^2 x - 1$

$$\begin{aligned} \cot(\cot^{-1} x) &= x \\ (-\cot^2(\cot^{-1} x) - 1) \cdot \frac{d}{dx}(\cot^{-1} x) &= 1 & \cot(\cot^{-1} x) = x \\ (-x^2 - 1) \cdot \frac{d}{dx}(\cot^{-1} x) &= 1 & \text{divide by } -x^2 - 1 \\ \frac{d}{dx}(\cot^{-1} x) &= \frac{1}{-x^2 - 1} = -\frac{1}{x^2 + 1} \end{aligned}$$

Theorem 11 and 12: $\frac{d}{dx} (\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2 - 1}}$ and $\frac{d}{dx} (\csc^{-1} x) = -\frac{1}{|x| \sqrt{x^2 - 1}}$

Proof: We compose the function $\sec x$ with its inverse $\sec^{-1} x$ and differentiate. Recall that $\frac{d}{dx} (\sec x) = \sec x \tan x$.

$$\begin{aligned} \sec(\sec^{-1} x) &= x \\ \sec(\sec^{-1} x) \tan(\sec^{-1} x) \cdot \frac{d}{dx}(\sec^{-1} x) &= 1 & \sec(\sec^{-1} x) = x \\ x \tan(\sec^{-1} x) \cdot \frac{d}{dx}(\sec^{-1} x) &= 1 & \text{divide by } x \tan(\sec^{-1} x) \\ \frac{d}{dx}(\sec^{-1} x) &= \frac{1}{x \tan(\sec^{-1} x)} \end{aligned}$$

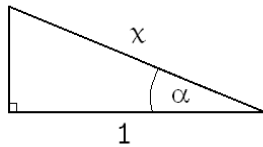
We now just need to simplify the expression $\tan(\sec^{-1} x)$.

Method 1. To simplify $\tan(\sec^{-1} x)$, we introduce a new variable α . Let $\alpha = \sec^{-1} x$. Then we have $\tan(\underbrace{\sec^{-1} x}_{\alpha}) = \tan \alpha$ where $\sec \alpha = x$ and α is between 0 and π . Recall that $\sec^2 \alpha = \tan^2 \alpha + 1$. If we don't have this formula memorized, we can easily derive it from the Pythagorean identity.

$$\begin{aligned} \sin^2 \alpha + \cos^2 \alpha &= 1 && \text{divide by } \cos^2 \alpha \\ \frac{\sin^2 \alpha}{\cos^2 \alpha} + \frac{\cos^2 \alpha}{\cos^2 \alpha} &= \frac{1}{\cos^2 \alpha} \\ \tan^2 \alpha + 1 &= \sec^2 \alpha \\ \tan \alpha &= \pm \sqrt{\sec^2 \alpha - 1} \end{aligned}$$

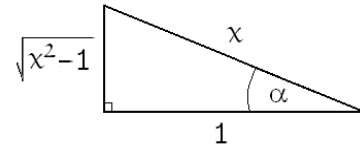
Method 2. To simplify $\tan(\sec^{-1} x)$, we introduce a new variable α . Let $\alpha = \sec^{-1} x$. Then $\sec \alpha = x$ and α is between 0 and π . Then we need to compute $\tan \alpha$.

We draw a right triangle in which $\sec \alpha = x = \frac{x}{1}$.



We find the missing side via the Pythagorean Theorem: $\sqrt{x^2 - 1}$.

Now we can read *any* trigonometric function value using this triangle.



Now we read from the triangle:

$$\tan \alpha = \tan(\sec^{-1} x) = \frac{\sqrt{x^2 - 1}}{1} = \sqrt{x^2 - 1}$$

The answer at this point is really $\pm \sqrt{x^2 - 1}$ as the triangle gave us the answer only up to a sign. Thus the derivative is

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x(\pm \sqrt{x^2 - 1})} = \pm \frac{1}{x\sqrt{x^2 - 1}}$$

We now need to figure out the sign of the derivative. From the graph of $\sec^{-1} x$ we can see that it is strictly increasing on both intervals making up its domain, thus the derivative is always positive. If x is positive, then $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$ and if x is negative, then $\frac{d}{dx}(\sec^{-1} x) = -\frac{1}{x\sqrt{x^2 - 1}}$. This can be expressed in a shorter form as

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

The proof for $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$ is virtually identical. As before, we compose the function $\csc x$ with its inverse and differentiate. Recall that $\frac{d}{dx}(\csc x) = -\csc x \cot x$

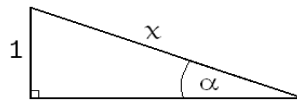
$$\begin{aligned} \csc(\csc^{-1} x) &= x \\ -\csc(\csc^{-1} x) \cot(\csc^{-1} x) \cdot \frac{d}{dx}(\csc^{-1} x) &= 1 && \csc(\csc^{-1} x) = x \\ -x \cot(\csc^{-1} x) \cdot \frac{d}{dx}(\csc^{-1} x) &= 1 && \text{divide by } -x \cot(\csc^{-1} x) \\ \frac{d}{dx}(\csc^{-1} x) &= -\frac{1}{x \cot(\csc^{-1} x)} \end{aligned}$$

We need to simplify $\cot(\csc^{-1} x)$. Let $\alpha = \csc^{-1} x$. $\cot(\underbrace{\csc^{-1} x}_{\alpha}) = \cot \alpha$ where $\csc \alpha = x$ and α is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

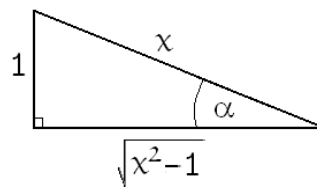
Method 1. We start with the Pythagorean identity and divide both sides by $\sin^2 \alpha$.

$$\begin{aligned}\sin^2 \alpha + \cos^2 \alpha &= 1 && \text{divide by } \sin^2 \alpha \\ \frac{\sin^2 \alpha}{\sin^2 \alpha} + \frac{\cos^2 \alpha}{\sin^2 \alpha} &= \frac{1}{\sin^2 \alpha} \\ 1 + \cot^2 \alpha &= \csc^2 \alpha \\ \cot^2 \alpha &= \csc^2 \alpha - 1 \\ \cot \alpha &= \pm \sqrt{\csc^2 \alpha - 1}\end{aligned}$$

Method 2. We draw a right triangle in which $\csc \alpha = x = \frac{x}{1}$.



We find the missing side using the Pythagorean Theorem and read the desired trigonometric function value.



Thus the derivative is

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x(\pm\sqrt{x^2-1})} = \pm \frac{1}{x\sqrt{x^2-1}}$$

From the graph of $\csc^{-1} x$ we can see that it is strictly decreasing on both intervals of its domain, thus the derivative is always negative. If x is positive, then $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$ and if x is negative, then $\frac{d}{dx}(\csc^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$. This can be expressed in a shorter form as

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$