Many experiments consist of repeating the same simple trial several times: we toss a coin 10 times, a basketball player takes 5 free throws, or a student guesses on every question of a quiz. In each trial, we only care whether something happens (a *success*) or not (a *failure*). In this handout we learn how to find the probability of getting exactly k successes.

Example 1. A fair coin is tossed three times. Find the probability of getting exactly two heads.

Solution. There are $2^3 = 8$ equally likely outcomes:

$$HHH \quad HHT \quad HTH \quad THH \quad HTT \quad THT \quad TTH \quad TTT$$

Three of them, HHT, HTH, and THH, have exactly two heads. So $P(\text{exactly two heads}) = \frac{3}{8}$.

Let us look at this answer in a different way. Each of the three good outcomes has probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$, and the number of good outcomes is the number of ways to choose which two of the three tosses are heads, $\binom{3}{2} = 3$. So the answer is $3 \cdot \frac{1}{8} = \frac{3}{8}$. This way of thinking also works when the outcomes are not equally likely.

Example 2. A fair die is rolled four times. Find the probability of getting exactly two 6s.

Solution. In each roll, a success is a 6, with probability $\frac{1}{6}$, and a failure is anything else, with probability $\frac{5}{6}$. Let us write S for success and F for failure. One way to get exactly two 6s is $SSFF$: the first two rolls are 6s, and the last two are not. The rolls are independent, so by the AND rule,

$$P(SSFF) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 = \frac{25}{1\,296}.$$

But the two 6s can come in any two of the four rolls. There are $\binom{4}{2} = 6$ ways to choose these two rolls:

$$SSFF \quad SFSF \quad SFFS \quad FSSF \quad FSFS \quad FFSS$$

Each of these has the same probability, $\frac{25}{1\,296}$, because each contains two factors of $\frac{1}{6}$ and two factors of $\frac{5}{6}$, only in a different order. These outcomes are mutually exclusive, so we add their probabilities:

$$P(\text{exactly two 6s}) = 6 \cdot \frac{25}{1\,296} = \frac{150}{1\,296} = \frac{25}{216} \approx 0.116$$

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The Binomial Probability Formula

The experiments in Examples 1 and 2 have the same structure.

Definition. A **binomial experiment** is an experiment with the following properties.

1. The same trial is repeated a fixed number of times, n.
2. Each trial has two possible results: success or failure.
3. The trials are independent.
4. The probability of success, p, is the same in every trial.

Repeating the reasoning of Example 2 in general gives the following formula.

Binomial Probability Formula. In a binomial experiment with n trials and probability of success p, the probability of exactly k successes is

$$P(\text{exactly } k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Here $\binom{n}{k}$ is the number of ways to choose which k of the n trials are successes, p^k is the probability of the k successes, and $(1-p)^{n-k}$ is the probability of the $n-k$ failures.

Example 3. A basketball player makes 80% of her free throws. She takes 5 free throws. Assume that her shots are independent. Find the probability that she makes

a) exactly 4 shots, b) at least 4 shots.

Solution. This is a binomial experiment with $n = 5$ and $p = 0.8$, so $1 - p = 0.2$.

a) We use the formula with $k = 4$.

$$P(\text{exactly } 4) = \binom{5}{4} (0.8)^4 (0.2)^1 = 5 \cdot 0.4096 \cdot 0.2 = 0.4096$$

b) "At least 4" means 4 or 5. These are mutually exclusive, so we add their probabilities. If she makes all 5 shots, then $k = 5$:

$$P(\text{exactly } 5) = \binom{5}{5} (0.8)^5 (0.2)^0 = 1 \cdot 0.32768 \cdot 1 = 0.32768$$

So $P(\text{at least } 4) = 0.4096 + 0.32768 = 0.73728 \approx 0.737$.

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Example 4. A student takes a quiz with 10 multiple-choice questions. Each question has 4 choices, and the student guesses on every question. Find the probability that the student gets

a) exactly 3 questions right, b) at least one question right.

Round your answers to three decimal places.

Solution. Each guess is right with probability $p = \frac{1}{4} = 0.25$, so $n = 10$ and $1 - p = 0.75$.

a) With $k = 3$,

$$P(\text{exactly } 3) = \binom{10}{3} (0.25)^3 (0.75)^7 = 120 \cdot 0.015625 \cdot 0.13348... \approx 0.250$$

b) The complement of "at least one right" is "none right," which means $k = 0$.

$$P(\text{at least one}) = 1 - \binom{10}{0} (0.25)^0 (0.75)^{10} = 1 - 0.75^{10} \approx 1 - 0.056 = 0.944$$

All the Probabilities at Once

Sometimes it is useful to compute the probability of every possible number of successes.

Example 5. A family has 4 children. Assume that each child is equally likely to be a boy or a girl. Find the probability of each possible number of girls, and use these to find the probability that the family has at least 2 girls.

Solution. Here $n = 4$ and $p = \frac{1}{2}$, so every term $p^k (1-p)^{n-k}$ is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$. Only the number of arrangements changes.

number of girls, k	0	1	2	3	4
$\binom{4}{k}$	1	4	6	4	1
probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

The probabilities add up to $\frac{16}{16} = 1$, as they should. The most likely result is 2 girls and 2 boys, but even that happens only $\frac{6}{16} = 37.5\%$ of the time.

$$P(\text{at least 2 girls}) = \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{11}{16} = 0.6875$$

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Example 6. At a store, 30% of the customers who try on a jacket end up buying it. Six customers try on the jacket. Find the probability that at least 2 of them buy it. Round your answer to three decimal places.

Solution. Here $n = 6$ and $p = 0.3$. "At least 2" means 2, 3, 4, 5, or 6, which is five cases. Its complement, "at most 1," has only two cases: 0 or 1.

$$P(0) = \binom{6}{0} (0.3)^0 (0.7)^6 = 0.117649 \qquad P(1) = \binom{6}{1} (0.3)^1 (0.7)^5 = 0.302526$$

So $P(\text{at least } 2) = 1 - 0.117649 - 0.302526 = 0.579825 \approx 0.580$.

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Practice Problems

1. A fair coin is tossed 5 times. Find the probability of getting

   a) exactly 3 heads, b) at least 4 heads, c) at least one head.

2. A fair die is rolled 3 times. Find the probability, rounded to three decimal places, of getting

   a) exactly one 6, b) at least one 6.

3. Decide whether each experiment is a binomial experiment. If it is, give n and p. If it is not, explain why.

   a) A die is rolled 10 times, and we count the number of 6s.

   b) Five cards are drawn from a deck without replacement, and we count the number of hearts.

   c) A coin is tossed until the first head appears, and we count the number of tosses.

4. A basketball player makes 70% of his free throws. He takes 6 free throws. Find the probability, rounded to three decimal places, that he makes

   a) exactly 5 shots, b) all 6 shots, c) at least 5 shots.

5. A student guesses on every question of a true-false quiz with 8 questions. Find the probability, rounded to three decimal places, that the student gets

   a) exactly 6 questions right, b) at least 6 questions right.

6. A medication cures 90% of the patients who take it. It is given to 10 patients. Find the probability, rounded to three decimal places, that

   a) all 10 are cured, b) exactly 9 are cured, c) at least 9 are cured.

7. In a city, 40% of the voters support a new park. A random sample of 15 voters is selected. Find the probability that exactly 6 of them support the park. Round your answer to three decimal places.

8. A family has 5 children. Assume that each child is equally likely to be a boy or a girl. Find the probability that the family has

   a) exactly 2 boys, b) at least one girl.

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Answers

1. a) $\binom{5}{3}\left(\frac{1}{2}\right)^5 = \frac{10}{32} = \frac{5}{16}$ b) $\frac{5+1}{32} = \frac{3}{16}$ c) $1 - \frac{1}{32} = \frac{31}{32}$

2. a) $\binom{3}{1} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2 = \frac{75}{216} = \frac{25}{72} \approx 0.347$ b) $1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216} \approx 0.421$

3. a) Binomial, with $n = 10$ and $p = \frac{1}{6}$.

   b) Not binomial: the cards are drawn without replacement, so the trials are not independent, and the probability of a heart changes from draw to draw.

   c) Not binomial: the number of trials is not fixed.

4. a) $\binom{6}{5}(0.7)^5(0.3) \approx 0.303$ b) $0.7^6 \approx 0.118$ c) $0.302526 + 0.117649 \approx 0.420$

5. a) $\binom{8}{6}\left(\frac{1}{2}\right)^8 = \frac{28}{256} \approx 0.109$ b) $\frac{28+8+1}{256} = \frac{37}{256} \approx 0.145$

6. a) $0.9^{10} \approx 0.349$ b) $\binom{10}{9}(0.9)^9(0.1) \approx 0.387$ c) $0.387420 + 0.348678 \approx 0.736$

7. $\binom{15}{6}(0.4)^6(0.6)^9 = 5\,005 \cdot 0.4^6 \cdot 0.6^9 \approx 0.207$

8. a) $\binom{5}{2}\left(\frac{1}{2}\right)^5 = \frac{10}{32} = \frac{5}{16}$ b) $1 - \left(\frac{1}{2}\right)^5 = \frac{31}{32}$

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