Powers of a Binomial

A **binomial** is a sum of two terms, such as $a + b$. Multiplying a binomial by itself repeatedly quickly becomes tedious:

$$\begin{aligned}(a+b)^0 &= 1,\\ (a+b)^1 &= a+b,\\ (a+b)^2 &= a^2+2ab+b^2,\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3,\\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4.\end{aligned}$$

Several patterns are visible in the expansion of $(a+b)^n$:

- There are $n+1$ terms.
- The exponent on a starts at n and decreases to 0; the exponent on b starts at 0 and increases to n. In every term the two exponents add up to n.
- The coefficients are symmetric: they read the same left to right as right to left.

The Binomial Theorem makes these observations exact and gives a formula for the coefficients.

Factorials and Binomial Coefficients

Definition: For a positive integer n, the **factorial** $n!$ is the product of the integers from 1 through n:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1.$$

By convention, $0! = 1$.

For instance $4! = 24$, $5! = 120$, and $6! = 720$. Factorials grow rapidly. A useful cancellation identity is $n! = n \cdot (n-1)!$, which lets large factorials in a quotient collapse:

$$\frac{10!}{8!} = \frac{10 \cdot 9 \cdot 8!}{8!} = 10 \cdot 9 = 90.$$

Definition: For integers $n \geq 0$ and $0 \leq k \leq n$, the **binomial coefficient** $\binom{n}{k}$ (read "n choose k") is

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

The number $\binom{n}{k}$ counts the ways to choose k objects from n without regard to order.

Theorem: (Properties of binomial coefficients) For integers $0 \le k \le n$:

$$\binom{n}{0} = \binom{n}{n} = 1, \qquad \binom{n}{k} = \binom{n}{n-k} \quad \text{(symmetry)}.$$

The symmetry says that choosing k objects out of n is the same as choosing the $n - k$ objects we leave out.

Example 1. Evaluate the given binomial coefficients. $\binom{7}{0}$ $\binom{7}{1}$ $\binom{7}{2}$ $\binom{7}{7}$ $\binom{10}{8}$

Solution: We don't always need a formula. $\binom{7}{0} = 1$, because the only 0-element subset of a 7-element set is the empty set. $\binom{7}{1} = 7$, because we can choose one element in 7 ways. $\binom{7}{7} = 1$, because the only 7-element subset is the whole set. For $\binom{7}{2}$ we use the formula, and for $\binom{10}{8}$ we use the symmetry first.

$$\binom{7}{2} = \frac{7 \cdot 6}{2!} = \frac{7 \cdot 6}{2 \cdot 1} = 21 \qquad \binom{10}{8} = \binom{10}{2} = \frac{10 \cdot 9}{2 \cdot 1} = 45$$

The Binomial Theorem

Theorem: (Binomial Theorem) For every non-negative integer n and all real (or complex) numbers a and b,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n.$$

Example 2. Expand $(x+1)^5$.

Solution. Here $a = x$, $b = 1$, and $n = 5$. The binomial coefficients are

$$\binom{5}{0} = 1, \quad \binom{5}{1} = 5, \quad \binom{5}{2} = \frac{5 \cdot 4}{2 \cdot 1} = 10, \quad \binom{5}{3} = \binom{5}{2} = 10, \quad \binom{5}{4} = \binom{5}{1} = 5, \quad \binom{5}{5} = 1.$$

Every power of 1 is 1, so

$$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1.$$

Why does the Binomial Theorem work? Consider

$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b).$$

Another way to apply the distributive law is this: from each of the four parentheses we choose either a or b, and multiply the four chosen letters. Then we repeat this in every possible way and add up all the products. Since there are two choices in each of the four parentheses, we get $2 \cdot 2 \cdot 2 \cdot 2 = 16$ products.

We can get a^4 in only one way: we pick a in every parentheses.

We get a^3b when we pick b in exactly one of the four parentheses. We can choose that parentheses in $\binom{4}{1} = 4$ ways, so the expansion contains $4a^3b$.

We get a^2b^2 when we pick b in exactly two of the four parentheses. These two parentheses can be chosen in $\binom{4}{2} = 6$ ways: $bbaa$, $baba$, $baab$, $abba$, $abab$, $aabb$. So the expansion contains $6a^2b^2$.

In the same way, ab^3 comes up $\binom{4}{3} = 4$ times, and b^4 comes up $\binom{4}{4} = 1$ time. Therefore

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

The coefficients add up to $1 + 4 + 6 + 4 + 1 = 16$, the number of all products. The same argument works for every n: the term $a^{n-k}b^k$ comes up as many times as we can choose the k parentheses where we pick b, and that is $\binom{n}{k}$.

Example 3. Expand $(x+3)^4$.

Solution. With $a = x$, $b = 3$, $n = 4$, and coefficients $1, 4, 6, 4, 1$:

$$\begin{aligned}(x+3)^4 &= x^4 + 4x^3(3) + 6x^2(3)^2 + 4x(3)^3 + (3)^4 \\ &= x^4 + 12x^3 + 54x^2 + 108x + 81.\end{aligned}$$

Example 4. Expand $(2y-1)^5$.

Solution. Write it as $(2y + (-1))^5$, so $a = 2y$ and $b = -1$. The binomial coefficients are $\binom{5}{0} = 1$, $\binom{5}{1} = 5$, $\binom{5}{2} = 10$, $\binom{5}{3} = 10$, $\binom{5}{4} = 5$, $\binom{5}{5} = 1$.

$$\begin{aligned}(2y-1)^5 &= (2y)^5 + 5(2y)^4(-1) + 10(2y)^3(-1)^2 + 10(2y)^2(-1)^3 + 5(2y)(-1)^4 + (-1)^5 \\ &= 32y^5 - 80y^4 + 80y^3 - 40y^2 + 10y - 1.\end{aligned}$$

Note how the signs alternate, a general feature of $(a-b)^n$.

Example 5. Compute 11^4 without a calculator.

Solution. We write $11 = 10 + 1$ and use the coefficients $1, 4, 6, 4, 1$. Notice that the digits of the result are exactly these coefficients.

$$11^4 = (10+1)^4 = 10^4 + 4 \cdot 10^3 + 6 \cdot 10^2 + 4 \cdot 10 + 1 = 10\,000 + 4\,000 + 600 + 40 + 1 = 14\,641$$

Finding a Single Term

Often only one term of an expansion is needed. In the Binomial Theorem, the term corresponding to index k is the $(k+1)$st term.

Theorem: (The General term) In the expansion of $(a+b)^n$, the $(k+1)$st term is

$$T_{k+1} = \binom{n}{k} a^{n-k} b^k, \qquad k = 0, 1, 2, \ldots, n.$$

Example 6. Find the 5th term of $(x+2)^9$.

Solution. The 5th term has $k+1 = 5$, so $k = 4$, with $n = 9$, $a = x$, $b = 2$:

$$T_5 = \binom{9}{4} x^{9-4}\, 2^4 = 126\, x^5\, (16) = 2016\, x^5.$$

Example 7. Find the coefficient of x^6 in the expansion of $\left(x^2 - \frac{1}{x}\right)^9$.

Solution: Here $a = x^2$, $b = -\frac{1}{x} = -x^{-1}$, and $n = 9$. The general term is

$$T_{k+1} = \binom{9}{k} (x^2)^{9-k}\, (-x^{-1})^k = \binom{9}{k} (-1)^k\, x^{2(9-k)}\, x^{-k} = \binom{9}{k} (-1)^k\, x^{18-3k}.$$

Set the exponent equal to 6:

$$18 - 3k = 6 \quad \Longrightarrow \quad k = 4.$$

The corresponding coefficient is

$$\binom{9}{4} (-1)^4 = 126.$$

So the coefficient of x^6 is 126.

Example 8. A coin is flipped 5 times. In how many different ways can we get exactly 2 heads?

Solution. Let h stand for heads and t for tails, and consider

$$(h+t)^5 = (h+t)(h+t)(h+t)(h+t)(h+t).$$

Each parentheses is one flip. When we pick h or t from each parentheses, we write down one possible sequence of flips. For example, picking h, t, t, h, t gives $htth$, which is h^2t^3. So the sequences with exactly 2 heads are the products equal to h^2t^3, and their number is the coefficient of h^2t^3:

$$\binom{5}{2} = \frac{5 \cdot 4}{2 \cdot 1} = 10.$$

There are 10 ways to get exactly 2 heads, out of $2^5 = 32$ possible sequences.



Enrichment

Sums of Binomial Coefficients

Let us add up the binomial coefficients in the expansion of $(a+b)^4$, first without and then with alternating signs.

$$\binom{4}{0}+\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4} = 1+4+6+4+1 = 16 = 2^4$$

$$\binom{4}{0}-\binom{4}{1}+\binom{4}{2}-\binom{4}{3}+\binom{4}{4} = 1-4+6-4+1 = 0$$

Let us do the same with $(a+b)^5$.

$$\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5} = 1+5+10+10+5+1 = 32 = 2^5$$

$$\binom{5}{0}-\binom{5}{1}+\binom{5}{2}-\binom{5}{3}+\binom{5}{4}-\binom{5}{5} = 1-5+10-10+5-1 = 0$$

This is not a coincidence. Choosing convenient values of a and b in the Binomial Theorem yields these results for every n.

Theorem: For every non-negative integer n,

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n} = 2^n,$$

and, for $n \geq 1$,

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots+(-1)^n\binom{n}{n} = 0.$$

Proof: For the first statement, we set $a = 1$ and $b = 1$ in the Binomial Theorem. Every power of 1 is 1, so

$$2^n = (1+1)^n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}.$$

For the second statement, we set $a = 1$ and $b = -1$. The powers of -1 alternate between 1 and -1, so

$$0 = 0^n = (1-1)^n = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots+(-1)^n\binom{n}{n}.$$

(We need $n \geq 1$ here, because $0^0 = 1$.)



Practice Problems

Factorials and binomial coefficients.

1. Evaluate: (a) $7!$ (b) $\dfrac{12!}{10!}$ (c) $\dfrac{8!}{3!\,5!}$.
2. Evaluate: (a) $\binom{10}{3}$ (b) $\binom{15}{15}$ (c) $\binom{9}{7}$ (d) $\binom{100}{1}$.
3. Use symmetry to evaluate $\binom{20}{18}$ without simplifying a large factorial from scratch.
4. Verify that $\binom{6}{2} + \binom{6}{3} = \binom{7}{3}$.

Expand each of the given expressions.

5. $(x+2)^5$
6. $(a-b)^6$
7. $(3x+1)^4$
8. $(2-x^2)^4$
9. $\left(x+\dfrac{1}{x}\right)^4$

Evaluate single terms and coefficients.

10. Find the 4th term of $(x+5)^7$.
11. Find the term containing a^3 in $(a+2b)^8$.
12. Find the middle term of $(2x-3)^6$.
13. Find the coefficient of x^5 in $(x-2)^9$.
14. Find the coefficient of x^4 in $\left(x^2+\dfrac{1}{x}\right)^8$.
15. Find the constant term (the term with no x) in $\left(x-\dfrac{2}{x}\right)^6$.



Answers

1. (a) $7! = 5040$. (b) $\dfrac{12!}{10!} = 12 \cdot 11 = 132$. (c) $\dfrac{8!}{3!\,5!} = \dfrac{8 \cdot 7 \cdot 6 \cdot 5!}{3! \cdot 5!} = \dfrac{8 \cdot 7 \cdot 6}{6} = 56$.

2. (a) $\binom{10}{3} = 120$. (b) $\binom{15}{15} = 1$. (c) $\binom{9}{7} = \binom{9}{2} = 36$. (d) $\binom{100}{1} = 100$.

3. $\binom{20}{18} = \binom{20}{2} = \dfrac{20 \cdot 19}{2} = 190$.

4. $\binom{6}{2} + \binom{6}{3} = 15 + 20 = 35 = \binom{7}{3}$.

5. $(x+2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$.

6. $(a-b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

7. $(3x+1)^4 = 81x^4 + 108x^3 + 54x^2 + 12x + 1$.

8. $(2-x^2)^4 = 16 - 32x^2 + 24x^4 - 8x^6 + x^8$.

9. $\left(x + \dfrac{1}{x}\right)^4 = x^4 + 4x^2 + 6 + \dfrac{4}{x^2} + \dfrac{1}{x^4}$.

10. $k+1 = 4 \Rightarrow k = 3$: $T_4 = \binom{7}{3} x^4 (5)^3 = 35 \cdot 125\, x^4 = 4375\, x^4$.

11. General term $\binom{8}{k} a^{8-k} (2b)^k$; need $8-k = 3$, so $k = 5$: $\binom{8}{5} a^3 (2b)^5 = 56 \cdot 32\, a^3 b^5 = 1792\, a^3 b^5$.

12. $n = 6$ has 7 terms; the middle one is the 4th, $k = 3$: $T_4 = \binom{6}{3} (2x)^3 (-3)^3 = 20 \cdot 8 \cdot (-27)\, x^3 = -4320\, x^3$.

13. General term $\binom{9}{k} x^{9-k} (-2)^k$; need $9-k = 5$, so $k = 4$: coefficient $= \binom{9}{4} (-2)^4 = 126 \cdot 16 = 2016$.

14. General term $\binom{8}{k} (x^2)^{8-k} (x^{-1})^k = \binom{8}{k} x^{16-3k}$; need $16 - 3k = 4$, so $k = 4$: coefficient $= \binom{8}{4} = 70$.

15. General term $\binom{6}{k} x^{6-k} \left(-\dfrac{2}{x}\right)^k = \binom{6}{k} (-2)^k x^{6-2k}$; need $6 - 2k = 0$, so $k = 3$: constant term $= \binom{6}{3} (-2)^3 = 20 \cdot (-8) = -160$.