

Example 1. We place \$3 000 in a bank account that pays 6% interest, compounded annually (the interest is added to the account once a year). How much money is in the account

- (a) after one year?
- (b) after 20 years?

Solution.

- (a) The interest is 6% of \$3 000, which is \$180, so the account holds $3\,000 + 180 = \$3\,180.00$. Equivalently, the balance is multiplied by 1.06.
- (b) Each year the balance is multiplied by 1.06, so after 20 years

$$A = 3\,000(1.06)^{20} \approx \$9\,621.41.$$

Another bank also offers 6% per year, but it pays 3% after six months and another 3% after twelve months, each time on the current balance. Interest is **compounded** each time it is added and starts earning interest itself. The first bank compounds once a year (**annually**); this bank compounds twice a year (**semiannually**).

Example 2. We place \$3 000 in a bank account with a 6% interest rate, compounded semiannually. How much money is in the account

- (a) after one year?
- (b) after 20 years?

Solution.

- (a) Each six months the balance is multiplied by 1.03, and in one year this happens twice:

$$A = 3\,000(1.03)(1.03) = 3\,000(1.03)^2 = \$3\,182.70.$$

- (b) In 20 years there are 40 six-month periods, so

$$A = 3\,000(1.03)^{40} \approx \$9\,786.11.$$

A third bank, also at 6% per year, adds interest every three months: it pays 1.5% four times a year. This is compounding **quarterly**.

Example 3. We place \$3 000 in a bank account with a 6% interest rate, compounded quarterly. How much money is in the account

- (a) after one year?
- (b) after 20 years?

Solution.

- (a) Each quarter the balance is multiplied by 1.015, and there are 4 quarters in a year:

$$A = 3\,000(1.015)^4 \approx \$3\,184.09.$$

- (b) In 20 years there are 80 quarters, so

$$A = 3\,000(1.015)^{80} \approx \$9\,871.99.$$

Many banks compound **monthly**: they pay $\frac{6\%}{12} = 0.5\%$ twelve times a year.

Example 4. We place \$3 000 in a bank account with a 6% interest rate, compounded monthly. How much money is in the account

- (a) after one year?
- (b) after 20 years?

Solution.

- (a) Each month the balance is multiplied by 1.005, and there are 12 months in a year:

$$A = 3\,000(1.005)^{12} \approx \$3\,185.03.$$

- (b) In 20 years there are 240 months, so

$$A = 3\,000(1.005)^{240} \approx \$9\,930.61.$$

Finally, some accounts compound **daily**. In this course we use a 360-day year (the *ordinary interest* convention used by many banks), so the bank pays $\frac{6\%}{360}$ of the balance every day.

Example 5. We place \$3 000 in a bank account with a 6% interest rate, compounded daily. How much money is in the account

- (a) after one year?
- (b) after 20 years?

Solution.

- (a) Each day the balance is multiplied by $1 + \frac{0.06}{360}$, and there are 360 days in a year:

$$A = 3\,000 \left(1 + \frac{0.06}{360}\right)^{360} \approx \$3\,185.49.$$

- (b) In 20 years there are $20 \cdot 360 = 7\,200$ days, so

$$A = 3\,000 \left(1 + \frac{0.06}{360}\right)^{7\,200} \approx \$9\,959.35.$$

Here is what we found for the same \$3 000 and the same 6% rate:

Compounded	Periods per year	After 1 year	After 20 years
annually	1	\$3 180.00	\$9 621.41
semiannually	2	\$3 182.70	\$9 786.11
quarterly	4	\$3 184.09	\$9 871.99
monthly	12	\$3 185.03	\$9 930.61
daily	360	\$3 185.49	\$9 959.35

More frequent compounding always earns more, but the extra gain becomes smaller and smaller. The pattern in all five examples is the same, and we can now state it as a formula.

The number of times per year that interest is added is called the **compounding frequency**, and we denote it by n . The annual rate r is called the **nominal rate**. Each time interest is added, the bank pays $\frac{r}{n}$ of the current balance, so the balance is multiplied by $1 + \frac{r}{n}$. In t years there are nt such periods.

Compound Interest Formula. If P dollars are invested at a nominal annual rate r , compounded n times per year, then the accumulated amount after t years is

$$A = P \left(1 + \frac{r}{n} \right)^{nt},$$

where r is written as a decimal and t is measured in years. The interest earned is $I = A - P$.

The usual compounding frequencies are annually ($n = 1$), semiannually ($n = 2$), quarterly ($n = 4$), monthly ($n = 12$), and daily ($n = 360$). When $n = 1$ this is the formula $A = P(1 + r)^t$ for annual compounding.

Rounding. Round money amounts to the nearest cent and percentages to the nearest tenth of a percent. Keep full calculator accuracy during a calculation, and round only the final answer.

Continuous Compounding

What if the interest is compounded every hour, every minute, every second? The balance keeps growing, but it never grows beyond a certain limit. This limiting case is called **continuous compounding**. It is described with the number $e \approx 2.71828$, which has its own key on a calculator.

Continuous Compounding. If P dollars are invested at a nominal annual rate r , compounded continuously, then the accumulated amount after t years is

$$A = Pe^{rt}.$$

Example 6. We place \$3 000 in a bank account with a 6% interest rate, compounded continuously. How much money is in the account

- (a) after one year?
- (b) after 20 years?

Solution. Here $P = 3\,000$ and $r = 0.06$.

- (a) $A = 3\,000 e^{0.06 \cdot 1} \approx \$3\,185.51$.
- (b) $A = 3\,000 e^{0.06 \cdot 20} = 3\,000 e^{1.2} \approx \$9\,960.35$.

This is only about \$1 more than daily compounding: after a certain point, compounding more often hardly matters.

The Effective Annual Rate

Because compounding more often earns more, a nominal rate does not tell the whole story. The **effective annual rate**, abbreviated EAR, is the annual rate which, compounded once a year, would give exactly the same balance after one year. It lets us compare accounts with different compounding frequencies.

Effective Annual Rate. If the nominal rate r is compounded n times per year, then

$$\text{EAR} = \left(1 + \frac{r}{n}\right)^n - 1.$$

Example 7. Bank A offers 6% compounded monthly. Bank B offers 6.1% compounded annually. Which account grows money faster?

Solution. The effective annual rate of Bank A is

$$\text{EAR} = \left(1 + \frac{0.06}{12}\right)^{12} - 1 \approx 1.0617 - 1 \approx 6.2\%.$$

Bank B compounds annually, so its effective annual rate is just its nominal rate, 6.1%. Since $6.2\% > 6.1\%$, Bank A pays more, in spite of its lower nominal rate, because of monthly compounding.



Practice Problems

Round money to the nearest cent and percentages to the nearest tenth of a percent.

- Find the accumulated value of \$4 000 invested at 6% for eight years if the interest is compounded
 - annually,
 - semiannually,
 - quarterly,
 - monthly,
 - daily,
 - continuously.
- How much will \$10 000 grow to in five years at 4.5% compounded quarterly?
- Find the effective annual rate corresponding to a nominal rate of 9% compounded
 - semiannually,
 - quarterly,
 - monthly.
- A credit card charges a nominal rate of 20% compounded monthly. Find its effective annual rate.
- Which is better: 7.9% compounded daily, or 8% compounded annually?
- How much money should we place in a bank account with an interest rate of 5%, compounded continuously, so that after 10 years there is \$10 000 in the account?



Answers

1. a) \$6 375.39 b) \$6 418.83 c) \$6 441.30 d) \$6 456.57 e) \$6 464.04 f) \$6 464.30

2. $A = 10\,000 \left(1 + \frac{0.045}{4}\right)^{20} \approx \$12\,507.51$

3. a) 9.2% b) 9.3% c) 9.4%

4. $\left(1 + \frac{0.2}{12}\right)^{12} - 1 \approx 21.9\%$

5. Compare the effective annual rates. For 7.9% compounded daily, $\left(1 + \frac{0.079}{360}\right)^{360} - 1 \approx 8.2\%$. The 8% account compounded annually has an effective annual rate of just 8%. So 7.9% compounded daily is better, in spite of the lower nominal rate.

6. We need $Pe^{0.05 \cdot 10} = 10\,000$, so $P = \frac{10\,000}{e^{0.5}} = 10\,000 e^{-0.5} \approx \6065.31 .