Let p and q be sentences, each with truth value true (T) or false (F). We use three connectives: negation ($\neg$, "not"), conjunction ($\wedge$, "and"), and disjunction ($\vee$, "or").

p	q	$\neg p$	$p \wedge q$	$p \vee q$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

$\neg p$ is the opposite of p. $p \wedge q$ is true only when both components are true. $p \vee q$ is true when at least one component is true.

Definition: Two sentences are **logically equivalent**, written $P \equiv Q$, if they have the same truth value in every possible case, that is, in every row of a truth table.

Before we state De Morgan's laws, let us see them in action.

Example 1. Negate the sentence "It is raining or it is cold."

Solution. Let p be "It is raining" and q be "It is cold." The sentence $p \vee q$ is true when at least one of its parts is true, so it is false exactly when neither part is true: it is not raining *and* it is not cold. The negation is "It is not raining *and* it is not cold." In symbols, $\neg(p \vee q)$ means the same as $\neg p \wedge \neg q$.

Example 2. Let p be "I have a cat" and q be "I have a dog." Four friends are talking about their pets. Ann has a cat and a dog, Ben has only a cat, Cora has only a dog, and Dan has neither. For each friend, decide whether the following sentences are true or false.

a) $\neg(p \vee q)$: "It is not true that I have a cat or a dog." and $\neg p \wedge \neg q$: "I don't have a cat and I don't have a dog."

b) $\neg(p \wedge q)$: "It is not true that I have a cat and a dog." and $\neg p \vee \neg q$: "I don't have a cat or I don't have a dog."

Solution. Only Dan can truthfully say that it is not true that he has a cat or a dog, and Dan is also the only one who has no cat and no dog. Only Ann has both pets, so she is the only one who cannot say "It is not true that I have a cat and a dog," and she is also the only one who cannot say "I don't have a cat or I don't have a dog."

	cat	dog	part a)		part b)	
	p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
Ann	T	T	F	F	F	F
Ben	T	F	F	F	T	T
Cora	F	T	F	F	T	T
Dan	F	F	T	T	T	T

In each part, the two sentences have the same truth value for every friend. The four friends cover all four possible cases, so this is not a coincidence: the two sentences in each part are logically equivalent.

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De Morgan's Laws. For any sentences p and q,

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \qquad \text{and} \qquad \neg(p \wedge q) \equiv \neg p \vee \neg q$$

To negate an "or" sentence, we negate each part and change "or" to "and". To negate an "and" sentence, we negate each part and change "and" to "or".

How Do We Prove an Equivalence?

Since p and q can only be true or false, there are just four possible cases. If we check all four, nothing is left out: **a truth table is a complete proof**. We compute both sides in every row and check that the two columns are identical.

Proof of $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$p \vee q$	$\boldsymbol{\neg(p \vee q)}$	$\neg p$	$\neg q$	$\boldsymbol{\neg p \wedge \neg q}$
T	T	T	**F**	F	F	**F**
T	F	T	**F**	F	T	**F**
F	T	T	**F**	T	F	**F**
F	F	F	**T**	T	T	**T**

The two bold columns agree in every row, so the two sentences are equivalent. ■

Proof of $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$p \wedge q$	$\boldsymbol{\neg(p \wedge q)}$	$\neg p$	$\neg q$	$\boldsymbol{\neg p \vee \neg q}$
T	T	T	**F**	F	F	**F**
T	F	F	**T**	F	T	**T**
F	T	F	**T**	T	F	**T**
F	F	F	**T**	T	T	**T**

Again, the two bold columns are identical. ■

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Connection to Sets

If $A = \{x \mid p(x)\}$ and $B = \{x \mid q(x)\}$, then the set operations complement, intersection, and union match the connectives $\neg$, $\wedge$, and $\vee$. The set laws $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$ are the same two laws written in different notation.

Example 3. Negate the sentence "$x > 2$ and $x \leq 7$."

Solution. By De Morgan's law, $\neg(p \wedge q) \equiv \neg p \vee \neg q$. The negation is "$x \leq 2$ *or* $x > 7$." Note that the negation is *not* "$x \leq 2$ and $x > 7$," which is impossible.

Simplifying Expressions

To simplify an expression, we push each negation inward one step at a time: negate each part and swap $\wedge$ and $\vee$. Then we use the double negation rule $\neg\neg p \equiv p$ to cancel pairs of negations. We repeat until every $\neg$ sits directly in front of a single letter.

Example 4. Simplify $\neg(p \vee \neg q)$.

Solution.

$$\begin{aligned} \neg(p \vee \neg q) &\equiv \neg p \wedge \neg\neg q && \text{De Morgan's law} \\ &\equiv \neg p \wedge q && \text{double negation} \end{aligned}$$

For example, "It is not true that I passed or I didn't study" means "I did not pass, and I did study."

Example 5. Simplify $\neg(\neg p \wedge \neg q)$.

Solution.

$$\begin{aligned} \neg(\neg p \wedge \neg q) &\equiv \neg\neg p \vee \neg\neg q && \text{De Morgan's law} \\ &\equiv p \vee q && \text{double negation} \end{aligned}$$

Example 6. Simplify $\neg(\neg p \wedge (q \vee \neg r))$.

Solution. We work from the outside in.

$$\begin{aligned} \neg(\neg p \wedge (q \vee \neg r)) &\equiv \neg\neg p \vee \neg(q \vee \neg r) && \text{De Morgan's law} \\ &\equiv p \vee (\neg q \wedge \neg\neg r) && \text{double negation, De Morgan's law} \\ &\equiv p \vee (\neg q \wedge r) && \text{double negation} \end{aligned}$$

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Example 7. Simplify $\neg(\neg(p \vee \neg q) \wedge r)$.

Solution. Here a negation sits inside another negation.

$$\begin{aligned} \neg(\neg(p \vee \neg q) \wedge r) &\equiv \neg\neg(p \vee \neg q) \vee \neg r && \text{De Morgan's law} \\ &\equiv (p \vee \neg q) \vee \neg r && \text{double negation} \\ &\equiv p \vee \neg q \vee \neg r \end{aligned}$$

Example 8. Show that $\neg(p \vee q)$ is *not* equivalent to $\neg p \vee \neg q$.

Solution. One row is enough. If p is T and q is F, then $\neg(p \vee q)$ is F, but $\neg p \vee \neg q$ is T. A single row where the two columns differ shows that the sentences are not equivalent.



Practice Problems

1. Negate each sentence in plain English, using De Morgan's laws.
   a) I will study math and I will study physics.
   b) The number is even or the number is prime.
   c) $x \geq 0$ and $x < 10$.
   d) The light is off or the door is locked.

2. Simplify each expression so that negation appears only directly in front of letters.

   a) $\neg(p \wedge \neg q)$ b) $\neg(\neg p \vee \neg q)$ c) $\neg(\neg p \vee q)$ d) $\neg(p \wedge (q \vee r))$

   e) $\neg(\neg p \vee (q \wedge \neg r))$ f) $\neg(\neg(\neg p \wedge q) \vee r)$ g) $\neg((p \vee \neg q) \wedge (\neg p \vee r))$ h) $\neg p \wedge \neg(\neg p \vee q)$

3. Prove $\neg(p \vee q) \equiv \neg p \wedge \neg q$ with a truth table of your own.

4. Use a truth table to prove that $\neg(p \wedge \neg q) \equiv \neg p \vee q$.

5. Prove the three-variable law $\neg(p \wedge q \wedge r) \equiv \neg p \vee \neg q \vee \neg r$ with a truth table. How many rows does the table need?

6. A student claims that $\neg(p \wedge q) \equiv \neg p \wedge \neg q$. Find a row of the truth table showing that the claim is false.

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Answers

1. a) I will not study math or I will not study physics.

   b) The number is not even and the number is not prime.

   c) $x < 0$ or $x \geq 10$.

   d) The light is on and the door is unlocked.

2. a) $\neg p \vee q$ b) $p \wedge q$ c) $p \wedge \neg q$ d) $\neg p \vee (\neg q \wedge \neg r)$ e) $p \wedge (\neg q \vee r)$

   f) $\neg p \wedge q \wedge \neg r$ g) $(\neg p \wedge q) \vee (p \wedge \neg r)$ h) $\neg p \wedge (p \wedge \neg q)$, which is always false, since p and $\neg p$ cannot both be true.

3. See the proof of $\neg(p \vee q) \equiv \neg p \wedge \neg q$ in this handout.

4. The two bold columns agree in every row.

p	q	$\neg q$	$p \wedge \neg q$	$\boldsymbol{\neg(p \wedge \neg q)}$	$\neg p$	$\boldsymbol{\neg p \vee q}$
T	T	F	F	**T**	F	**T**
T	F	T	T	**F**	F	**F**
F	T	F	F	**T**	T	**T**
F	F	T	F	**T**	T	**T**

5. The table needs $2^3 = 8$ rows. The left side $\neg(p \wedge q \wedge r)$ is F only in the row where p, q, and r are all T, and it is T in the other seven rows. The right side $\neg p \vee \neg q \vee \neg r$ is also F only when $\neg p$, $\neg q$, and $\neg r$ are all F, that is, when p, q, and r are all T. The two columns agree in all 8 rows.

6. If p is T and q is F, then $\neg(p \wedge q)$ is T, but $\neg p \wedge \neg q$ is F.

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