Throughout this handout, all sets are subsets of a universal set U. The three basic set operations are union, intersection, and complement. De Morgan's laws describe how the complement interacts with union and intersection.

Definition: For sets A and B,

$$\begin{aligned} A \cup B &= \{x \mid x \in A \text{ or } x \in B\} && \text{(union)} \\ A \cap B &= \{x \mid x \in A \text{ and } x \in B\} && \text{(intersection)} \\ \overline{A} &= \{x \in U \mid x \notin A\} && \text{(complement)} \end{aligned}$$

Before we state De Morgan's laws, let us see them in action.

Example 1. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, let A be the set of even numbers in U, and let B be the set of multiples of 3 in U. So $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 6, 9\}$.

a) Find $\overline{A \cup B}$ and $\overline{A} \cap \overline{B}$. b) Find $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$.

Solution. First we find the complements of the two sets: $\overline{A} = \{1, 3, 5, 7, 9\}$ and $\overline{B} = \{1, 2, 4, 5, 7, 8, 10\}$.

a) $A \cup B = \{2, 3, 4, 6, 8, 9, 10\}$, so $\overline{A \cup B} = \{1, 5, 7\}$. These are the numbers that are neither even nor multiples of 3.

The elements in both $\overline{A}$ and $\overline{B}$ are $\overline{A} \cap \overline{B} = \{1, 5, 7\}$. We got the same set!

b) $A \cap B = \{6\}$, so $\overline{A \cap B} = \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$.

The elements in at least one of $\overline{A}$ and $\overline{B}$ are $\overline{A} \cup \overline{B} = \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$. Again, we got the same set!

This is not a coincidence. The same thing happens for any two sets.

De Morgan's Laws. For any sets A and B in a universal set U,

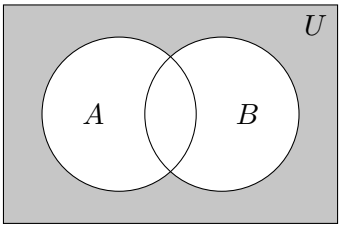
$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \text{and} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

In words: the complement of a union is the intersection of the complements, and the complement of an intersection is the union of the complements. An element is *not in A or B* exactly when it is *not in A and not in B*. An element is *not in both* sets exactly when it is *missing from at least one* of them.

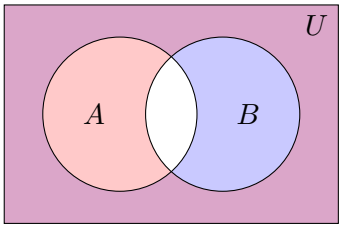
© Hidegkuti, 2026

Seeing the Laws with Venn Diagrams

The law $\overline{A \cup B} = \overline{A} \cap \overline{B}$. On the left, we shade $\overline{A \cup B}$: everything *outside* both circles. On the right, we shade $\overline{A}$ in blue and $\overline{B}$ in red. The purple region, where the two shadings overlap, is $\overline{A} \cap \overline{B}$. It is the same region.

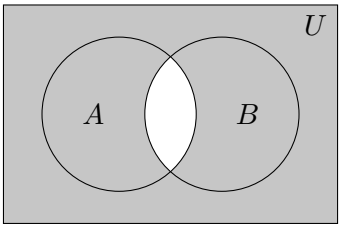


$\overline{A \cup B}$ is shaded

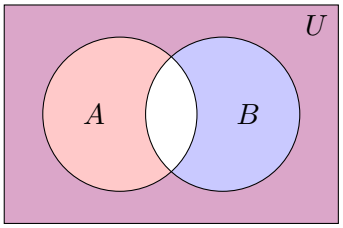


$\overline{A} \cap \overline{B}$ is purple

The law $\overline{A \cap B} = \overline{A} \cup \overline{B}$. On the left, $A \cap B$ is the lens-shaped region in the middle, so $\overline{A \cap B}$ is everything *except* the lens. On the right, the union $\overline{A} \cup \overline{B}$ is every region that has *at least one* color: again, everything except the lens.



$\overline{A \cap B}$ is shaded



$\overline{A} \cup \overline{B}$ is colored

A Venn diagram is a convincing picture, but it is not a proof. To *prove* that two sets are equal, we use the following principle.

> **Double Inclusion.** Two sets X and Y are equal if and only if $X \subseteq Y$ and $Y \subseteq X$.

To show that $X \subseteq Y$, we take an arbitrary element $x \in X$ and show that $x \in Y$.

Proof of $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Step 1. We show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$, so x is in neither A nor B. (If x were in A, it would be in $A \cup B$; the same is true for B.) Thus $x \notin A$ and $x \notin B$, that is, $x \in \overline{A}$ and $x \in \overline{B}$. Therefore $x \in \overline{A} \cap \overline{B}$.

Step 2. We show that $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. Let $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$, so $x \notin A$ and $x \notin B$. Since x belongs to neither set, it cannot belong to $A \cup B$. Hence $x \notin A \cup B$, so $x \in \overline{A \cup B}$.

By Steps 1 and 2 and double inclusion, $\overline{A \cup B} = \overline{A} \cap \overline{B}$. ■

© Hidegkuti, 2026

Proof of $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Step 1. We show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Let $x \in \overline{A \cap B}$. Then $x \notin A \cap B$, so x is not in both A and B. Hence $x \notin A$ or $x \notin B$. If $x \notin A$, then $x \in \overline{A}$; if $x \notin B$, then $x \in \overline{B}$. In either case, $x \in \overline{A} \cup \overline{B}$.

Step 2. We show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Let $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$. If $x \in \overline{A}$, then $x \notin A$, so $x \notin A \cap B$. If $x \in \overline{B}$, then $x \notin B$, so $x \notin A \cap B$. In both cases $x \notin A \cap B$, so $x \in \overline{A \cap B}$.

By Steps 1 and 2 and double inclusion, $\overline{A \cap B} = \overline{A} \cup \overline{B}$. ■

Example 2. Let U be the set of all integers, E the set of even integers, and T the set of multiples of 3. Describe the set $\overline{E \cup T}$ in words.

Solution. By De Morgan's law, $\overline{E \cup T} = \overline{E} \cap \overline{T}$. This is the set of integers that are odd *and* not divisible by 3, such as ± 1, ± 5, ± 7, ± 11, and so on.

Simplifying Expressions

To simplify an expression with a long complement bar, we push the bar inward one step at a time: break the bar, swap $\cup$ and $\cap$, and put a bar on each piece. Then we use the double complement rule $\overline{\overline{X}} = X$ (the complement of a complement is the original set) to remove double bars. We repeat until bars sit only on single letters.

Example 3. Simplify $\overline{A \cup \overline{B}}$.

Solution.

$$\begin{aligned} \overline{A \cup \overline{B}} &= \overline{A} \cap \overline{\overline{B}} && \text{De Morgan's law} \\ &= \overline{A} \cap B && \text{double complement} \end{aligned}$$

In words: the elements that are *not* in "A or outside B" are those that are outside A but inside B.

Example 4. Simplify $\overline{\overline{A} \cap \overline{B}}$.

Solution.

$$\begin{aligned} \overline{\overline{A} \cap \overline{B}} &= \overline{\overline{A}} \cup \overline{\overline{B}} && \text{De Morgan's law} \\ &= A \cup B && \text{double complement} \end{aligned}$$

© Hidegkuti, 2026

Example 5. Simplify $\overline{\overline{A} \cap (B \cup \overline{C})}$.

Solution. We work from the outside in.

$$\begin{aligned} \overline{\overline{A} \cap (B \cup \overline{C})} &= \overline{\overline{A}} \cup \overline{B \cup \overline{C}} && \text{De Morgan's law} \\ &= A \cup \left(\overline{B} \cap \overline{\overline{C}}\right) && \text{double complement, De Morgan's law} \\ &= A \cup (\overline{B} \cap C) && \text{double complement} \end{aligned}$$

Example 6. Simplify $\overline{\overline{A \cup \overline{B}} \cap C}$.

Solution. Here a bar sits inside another bar.

$$\begin{aligned} \overline{\overline{A \cup \overline{B}} \cap C} &= \overline{\overline{A \cup \overline{B}} \cup \overline{C}} && \text{De Morgan's law} \\ &= (A \cup \overline{B}) \cup \overline{C} && \text{double complement} \\ &= A \cup \overline{B} \cup \overline{C} \end{aligned}$$



Practice Problems

1. Let $U = \{a, b, c, d, e, f, g\}$, $A = \{a, b, c\}$, and $B = \{c, d, e\}$. Find each set.
   a) $\overline{A \cup B}$ b) $\overline{A} \cap \overline{B}$ c) $\overline{A \cap B}$ d) $\overline{A} \cup \overline{B}$
2. Draw Venn diagrams to illustrate that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
3. Let $U = \{1, 2, 3, \ldots, 20\}$, let A be the set of multiples of 2 in U, and let B be the set of multiples of 5 in U. List the elements of $\overline{A \cup B}$, and describe the set in words.
4. Simplify each expression using De Morgan's laws, so that complement bars sit only on single letters.
   a) $\overline{A \cap \overline{B}}$ b) $\overline{\overline{A} \cup \overline{B}}$ c) $\overline{\overline{A} \cup B}$ d) $\overline{A \cap (B \cup C)}$
   e) $\overline{\overline{A} \cup (B \cap \overline{C})}$ f) $\overline{\overline{\overline{A} \cap B} \cup C}$ g) $\overline{(A \cup \overline{B}) \cap (\overline{A} \cup C)}$ h) $\overline{A} \cap \overline{\overline{A} \cup B}$
5. Prove the three-set version $\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$ by showing that each side is a subset of the other.
6. True or false? $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for all sets A and B. If false, give a counterexample.

© Hidegkuti, 2026



Answers

1. a) $\{f, g\}$ b) $\{f, g\}$ c) $\{a, b, d, e, f, g\}$ d) $\{a, b, d, e, f, g\}$

2. In both diagrams, everything except the lens-shaped region $A \cap B$ is shaded.

3. $\overline{A \cup B} = \{1, 3, 7, 9, 11, 13, 17, 19\}$: the numbers in U that are divisible by neither 2 nor 5 (the odd numbers that do not end in 5).

4. a) $\overline{A} \cup B$ b) $A \cap B$ c) $A \cap \overline{B}$ d) $\overline{A} \cup (\overline{B} \cap \overline{C})$ e) $A \cap (\overline{B} \cup C)$
   f) $\overline{A} \cap B \cap \overline{C}$ g) $(\overline{A} \cap B) \cup (A \cap \overline{C})$ h) $\overline{A} \cap (A \cap \overline{B}) = \varnothing$, since no element is both in A and outside A.

5. Let $x \in \overline{A \cup B \cup C}$. Then $x \notin A \cup B \cup C$, so x is in none of A, B, C. Thus $x \in \overline{A}$, $x \in \overline{B}$, and $x \in \overline{C}$, so $x \in \overline{A} \cap \overline{B} \cap \overline{C}$. Conversely, let $x \in \overline{A} \cap \overline{B} \cap \overline{C}$. Then $x \notin A$, $x \notin B$, and $x \notin C$, so x cannot be in $A \cup B \cup C$. Hence $x \in \overline{A \cup B \cup C}$. By double inclusion, the two sets are equal.

6. False. Let $U = \{1, 2\}$, $A = \{1\}$, and $B = \{2\}$. Then $\overline{A \cup B} = \varnothing$, but $\overline{A} \cup \overline{B} = \{2\} \cup \{1\} = \{1, 2\}$.

For more documents like this, visit our page at https://teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.

© Hidegkuti, 2026