

Suppose we play a simple game: we roll a die, and we win as many dollars as the number that comes up. We cannot predict what will happen in a single game, but what happens if we play many times? Suppose we play 600 games. We expect each number to come up about 100 times, so we expect to win about

$$100 \cdot 1 + 100 \cdot 2 + 100 \cdot 3 + 100 \cdot 4 + 100 \cdot 5 + 100 \cdot 6 = \$2\,100$$

in all, which is $\frac{2\,100}{600} = \$3.50$ per game. Dividing by 600 is the same as multiplying each prize by its probability:

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5.$$

This number is the *average winning per game in the long run*. We call it the expected value of the game.

Definition. Suppose an experiment has outcomes with values $x_1, x_2, \dots, x_k$ (for example, the amounts of money we win or lose), and the probabilities of these outcomes are $p_1, p_2, \dots, p_k$. The **expected value** of the experiment is

$$E = x_1p_1 + x_2p_2 + \dots + x_kp_k.$$

That is, we multiply each value by its probability and add the products.

The expected value is a long-run average. It does not have to be a number that can actually occur: no single roll of a die ever gives 3.5. It tells us what to expect *on average* if the experiment is repeated many times.

Example 1. A raffle sells 1 000 tickets for \$2 each. There is one prize of \$500. We buy one ticket. What is the expected value of our net gain?

Solution. Our net gain is the prize minus the price of the ticket. If our ticket wins, we gain $500 - 2 = \$498$, and the probability of this is $\frac{1}{1\,000}$. Otherwise we lose the \$2 we paid, a net gain of $-\$2$, with probability $\frac{999}{1\,000}$.

Outcome	Net gain x	Probability p	$x \cdot p$
our ticket wins	\$498	$\frac{1}{1\,000}$	\$0.498
our ticket loses	-\$2	$\frac{999}{1\,000}$	-\$1.998

The expected value is the sum of the last column:

$$E = 0.498 - 1.998 = -\$1.50.$$

On average we lose \$1.50 on each ticket we buy.

Example 2. An American roulette wheel has 38 slots: the numbers 1 through 36, of which 18 are red, and two green slots, 0 and 00. We bet \$1 on red. If the ball lands on red we win \$1, otherwise we lose our \$1. What is the expected value of our net gain?

Solution. The probability of winning is $\frac{18}{38}$ and the probability of losing is $\frac{20}{38}$, so

$$E = 1 \cdot \frac{18}{38} + (-1) \cdot \frac{20}{38} = -\frac{2}{38} \approx -\$0.0526.$$

On average we lose about 5.3 cents on every dollar we bet. After 1 000 such bets we expect to be down about \$52.63. This is how the casino makes its money: the game is slightly in its favor, and over thousands of bets the average becomes very reliable.

Example 3. A game costs nothing to play. We roll two dice. If the sum is 7 we win \$6. If the sum is 2, 6, 8, or 12 we win \$3. Otherwise we win nothing. What is the expected value of our winnings?

Solution. There are 36 equally likely outcomes. A sum of 7 can occur in 6 ways. The sums 2, 6, 8, 12 can occur in $1 + 5 + 5 + 1 = 12$ ways. The remaining $36 - 6 - 12 = 18$ outcomes win nothing.

Sum	Winnings x	Probability p	$x \cdot p$
7	\$6	$\frac{6}{36}$	\$1
2, 6, 8, or 12	\$3	$\frac{12}{36}$	\$1
anything else	\$0	$\frac{18}{36}$	\$0

The expected value is $E = 1 + 1 + 0 = \$2$.

Fair Games

In the last example, the game was free, and we expect to win \$2 per game. Suppose now that the organizer charges a fee to play. A game is called **fair** if the expected value of the net gain is 0: in the long run, neither the player nor the organizer comes out ahead.

Fair price. The price that makes a game fair is the expected value of the winnings (the amount we receive, before subtracting the price).

So the fair price for the dice game is \$2. If the organizer charges \$3 to play, the expected net gain is $2 - 3 = -\$1$ per game, and the organizer expects to make \$1 per game. The raffle and the roulette wheel were not fair games: they were designed so that the organizer wins on average.

Example 4. An insurance company sells one-year life insurance policies. A policy pays \$50 000 to the family if the customer dies during the year, and costs the customer \$300. Based on past data, the probability that a customer of this age dies during the year is 0.004.

- (a) What is the expected value of the company's gain from one policy?
- (b) The company sells 10 000 such policies. How much profit does it expect?
- (c) What is the expected value of the customer's net gain?

Solution.

- (a) With probability 0.996 the customer survives, and the company keeps the \$300. With probability 0.004 the customer dies, and the company receives \$300 but pays \$50 000, a net gain of $300 - 50\,000 = -\$49\,700$.

$$E = 300(0.996) + (-49\,700)(0.004) = 298.80 - 198.80 = \$100.$$

The company expects to gain \$100 per policy.

- (b) $10\,000 \cdot \$100 = \$1\,000\,000$. The company cannot know which customers will die, but with so many policies the average is very predictable.
- (c) The customer's net gain is the opposite of the company's: the expected value is $-\$100$.

Why would anyone buy a policy with a negative expected value? Because the expected value is not the whole story. The customer is not paying for an average: the customer is protecting the family against a very large loss that would be a disaster. The same is true of car insurance and health insurance. The fair price of this policy would be $50\,000 \cdot 0.004 = \$200$, and the extra \$100 pays for the company's costs and profit.

Example 5. We can choose one of two offers. Offer A is a sure \$50. Offer B is a 10% chance to win \$600 (and nothing otherwise). Which offer has the greater expected value?

Solution. Offer A is certain, so its expected value is \$50. For offer B,

$$E = 600(0.10) + 0(0.90) = \$60.$$

Offer B has the greater expected value, so if we could play many times, B would be better. If we can play only once, the choice also depends on how much we need the \$50 for sure.



Practice Problems

Round money to the nearest cent.

1. A die is rolled. If the number is even we win that many dollars. If it is odd we lose \$3. Find the expected value of our net gain.
2. A raffle sells 500 tickets for \$5 each. There is one prize of \$1 000 and two prizes of \$200. We buy one ticket. What is the expected value of our net gain?
3. We toss three coins and win \$1 for each head. Find the expected value of the winnings, and the fair price of this game.
4. On an American roulette wheel, a bet of \$1 on a single number pays \$35 if that number comes up (we also keep our \$1), and we lose our \$1 otherwise. There are 38 equally likely slots. Find the expected value of the net gain, and compare it with the bet on red in the examples.
5. A European roulette wheel has only 37 slots: the numbers 1 through 36 (18 of them red) and a single green 0. We bet \$100 on red. Find the expected value of the net gain under the European rules, and under the American rules (38 slots). On which wheel would we rather play?
6. An insurance company sells one-year \$100 000 life insurance policies for \$500 each. The probability that a customer dies during the year is 0.003.
 - (a) What is the expected gain of the company per policy?
 - (b) The company sells 20 000 such policies. What profit does it expect?
 - (c) What would be the fair price of a policy?
7. One card is drawn from a standard deck of 52 cards. An ace pays \$10, a jack, queen, or king pays \$5, and any other card pays nothing. What is the fair price of this game?
8. We throw two dice. If the two numbers are the same, we win the sum of the numbers rolled (in dollars). Otherwise, we pay \$2. Should we play this game?
9. Offer A is a sure \$200. Offer B is a 25% chance to win \$1 000. Which has the greater expected value?
10. A multiple-choice test question has five choices. A correct answer scores 4 points and a wrong answer loses 1 point.
 - (a) A student guesses at random. What is the expected score on the question?
 - (b) The student can eliminate one wrong choice and guesses among the other four. What is the expected score now?



Answers

1. $E = \frac{2+4+6}{6} + (-3) \cdot \frac{3}{6} = 2 - 1.5 = \$0.50.$
2. We gain $1\,000 - 5 = \$995$ with probability $\frac{1}{500}$, $200 - 5 = \$195$ with probability $\frac{2}{500}$, and lose \$5 with probability $\frac{497}{500}$: $E = \frac{995 + 2 \cdot 195 - 5 \cdot 497}{500} = \frac{-1\,100}{500} = -\$2.20.$
3. The numbers of heads 0, 1, 2, 3 have probabilities $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$, so $E = \frac{0+3+6+3}{8} = \$1.50.$ The fair price is \$1.50.
4. $E = 35 \cdot \frac{1}{38} - 1 \cdot \frac{37}{38} = -\frac{2}{38} \approx -\$0.0526.$ This is exactly the same as the bet on red: on average the player loses about 5.3 cents per dollar either way.
5. European: we win \$100 with probability $\frac{18}{37}$ and lose \$100 with probability $\frac{19}{37}$, so $E = 100 \cdot \frac{18}{37} - 100 \cdot \frac{19}{37} = -\frac{100}{37} \approx -\$2.70.$ American: $E = 100 \cdot \frac{18}{38} - 100 \cdot \frac{20}{38} = -\frac{200}{38} \approx -\$5.26.$ The European wheel is better: the second green slot roughly doubles the house's edge.
6. a) $E = 500 - 100\,000(0.003) = 500 - 300 = \200 per policy. b) $20\,000 \cdot 200 = \$4\,000\,000.$ c) The fair price is $100\,000 \cdot 0.003 = \$300.$
7. $E = 10 \cdot \frac{4}{52} + 5 \cdot \frac{12}{52} = \frac{100}{52} \approx \$1.92.$ The fair price is about \$1.92.
8. The sums for doubles are 2, 4, 6, 8, 10, 12, which add up to 42, each with probability $\frac{1}{36}$. The other 30 outcomes cost \$2 each: $E = \frac{42}{36} + (-2) \cdot \frac{30}{36} = \frac{42-60}{36} = -\$0.50.$ The expected value is negative, so we should not play: on average we lose 50 cents per game.
9. The expected value of B is $1\,000(0.25) = \$250,$ which is greater than \$200, so B has the greater expected value.
10. a) $4 \cdot \frac{1}{5} + (-1) \cdot \frac{4}{5} = 0$ points: guessing neither helps nor hurts on average. b) $4 \cdot \frac{1}{4} + (-1) \cdot \frac{3}{4} = 0.25$ points: eliminating a choice makes guessing worthwhile.