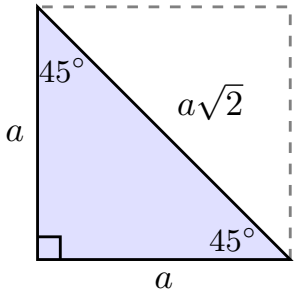
A few right triangles show up again and again in geometry problems and puzzles. They are so common that it is worth knowing them by heart. We call them the *famous triangles*. To understand them, all we need is the Pythagorean theorem and the fact that the angles of a triangle add up to 180°.

The isosceles right triangle

If we cut a square in half along one of its diagonals, we get two identical right triangles. Each of them is isosceles: its two legs are sides of the square, so they are equally long. The two acute angles are also equal, and since they add up to 90°, each of them is 45°. This triangle is called the **isosceles right triangle**, or the 45°-45°-90° triangle.

If the legs are a units long, the Pythagorean theorem gives us the hypotenuse c.

$c^2 = a^2 + a^2 = 2a^2$ so $c = \sqrt{2a^2} = a\sqrt{2}$



The isosceles right triangle: If the legs of an isosceles right triangle are a units long, then its hypotenuse is $a\sqrt{2}$ units long. In other words, the hypotenuse is $\sqrt{2}$ times as long as a leg.

Example 1. Find the missing sides of the isosceles right triangle.

a) The legs are 7 cm long. b) The hypotenuse is 10 in long.

Solution: a) The hypotenuse is $\sqrt{2}$ times as long as a leg, so the hypotenuse is $\boxed{7\sqrt{2}\,\text{cm}}$.

b) Let a be the length of the legs. By the Pythagorean theorem,

$a^2 + a^2 = (10\,\text{in})^2$

$2a^2 = 100\,\text{in}^2$

$a^2 = 50\,\text{in}^2$

$a = \sqrt{50}\,\text{in} = \sqrt{25 \cdot 2}\,\text{in} = 5\sqrt{2}\,\text{in}$

Each leg is $\boxed{5\sqrt{2}\,\text{in}}$ long. (We could also divide: $\dfrac{10}{\sqrt{2}} = \dfrac{10\sqrt{2}}{2} = 5\sqrt{2}$.)

Example 2. Find the length of the diagonal of a square with sides 6 m long.

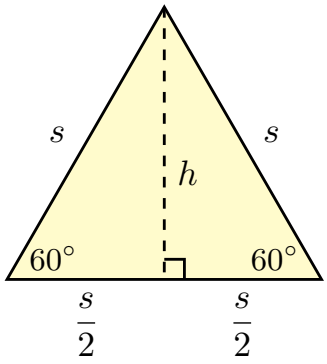
Solution: The diagonal cuts the square into two isosceles right triangles with legs 6 m. The diagonal is the hypotenuse, so it is $\boxed{6\sqrt{2}\,\text{m}}$ long.

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The regular triangle

A triangle with three equal sides is called a **regular** (or equilateral) triangle. Its three angles are also equal, so each of them is $\frac{180^\circ}{3} = 60^\circ$.
The height drawn from the top vertex cuts the regular triangle into two identical halves: one is the mirror image of the other. So the height meets the base at its midpoint, and it cuts the base into two segments of length $\frac{s}{2}$, where s is the length of a side. By the Pythagorean theorem,

$$h^2 + \left(\frac{s}{2}\right)^2 = s^2 \qquad \text{so} \qquad h^2 = s^2 - \frac{s^2}{4} = \frac{3s^2}{4} \qquad \text{and} \qquad h = \frac{\sqrt{3}}{2}s$$



The regular triangle: If the sides of a regular triangle are s units long, then its height is $h = \frac{\sqrt{3}}{2}s$ units, and its area is $A = \frac{1}{2} \cdot s \cdot \frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{4}s^2$ square units.

Example 3. Find the exact height and area of a regular triangle with sides 10 cm long.

Solution: The height cuts the base into two segments of 5 cm. By the Pythagorean theorem,

$$h^2 + (5\,\text{cm})^2 = (10\,\text{cm})^2$$

$$h^2 = 100\,\text{cm}^2 - 25\,\text{cm}^2 = 75\,\text{cm}^2$$

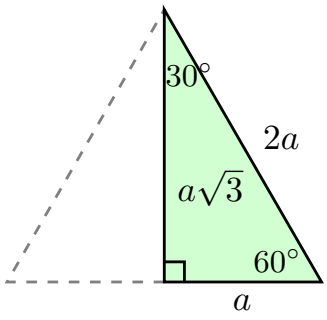
$$h = \sqrt{75}\,\text{cm} = \sqrt{25 \cdot 3}\,\text{cm} = 5\sqrt{3}\,\text{cm}$$

The height is $\boxed{5\sqrt{3}\,\text{cm}}$, and the area is $A = \frac{1}{2} \cdot 10\,\text{cm} \cdot 5\sqrt{3}\,\text{cm} = \boxed{25\sqrt{3}\,\text{cm}^2}$.

The semi-regular triangle

The height of a regular triangle cuts it into two identical right triangles. Each of them has a 60° angle (an angle of the regular triangle) and a 30° angle (half of the 60° angle at the top). We call this right triangle the **semi-regular triangle**, because it is half of a regular triangle. (This is the standard name in Hungary, but not in English. In American books, you will find it as the 30°-60°-90° triangle.) Its hypotenuse is a side of the regular triangle, and its shorter leg is half of a side. So if the shorter leg is a, then the hypotenuse is $2a$. The longer leg is the height of the regular triangle, so by the Pythagorean theorem,

$(2a)^2 = a^2 + x^2$ so $x^2 = 4a^2 - a^2 = 3a^2$ and $x = a\sqrt{3}$



The semi-regular triangle: In a right triangle with angles 30°, 60°, and 90°, the side opposite the 30° angle is half of the hypotenuse. If the shorter leg is a units long, then the hypotenuse is $2a$ units and the longer leg is $a\sqrt{3}$ units long.

Example 4. Find the missing sides of the semi-regular triangle.

a) The hypotenuse is 12 units. b) The shorter leg is 5 units. c) The longer leg is 9 units.

Solution: a) The shorter leg is half of the hypotenuse: $\boxed{6 \text{ units}}$. The longer leg is $\sqrt{3}$ times the shorter leg: $\boxed{6\sqrt{3} \text{ units}}$.

b) The hypotenuse is twice the shorter leg: $\boxed{10 \text{ units}}$, and the longer leg is $\boxed{5\sqrt{3} \text{ units}}$.

c) Let a be the shorter leg. Then the longer leg is $a\sqrt{3} = 9$, so

$$a = \frac{9}{\sqrt{3}} = \frac{9\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{9\sqrt{3}}{3} = 3\sqrt{3}$$

The shorter leg is $\boxed{3\sqrt{3} \text{ units}}$, and the hypotenuse is $2 \cdot 3\sqrt{3} = \boxed{6\sqrt{3} \text{ units}}$.

The statement in the box also works backwards.

Theorem: If one leg of a right triangle is half of the hypotenuse, then the angle opposite this leg is 30°, and the triangle is a semi-regular triangle.

To see why, let the leg be a and the hypotenuse $2a$, and reflect the triangle across its other leg (the dashed half in the picture above). The triangle and its mirror image together form a large triangle whose sides are $2a$, $2a$, and $a + a = 2a$. So the large triangle is regular, all of its angles are 60°, and the angle at the top, which is cut in half, is 30° in each half.

Example 5. An 8 m long ladder leans against a vertical wall. Its foot is 4 m away from the wall. What angle does the ladder make with the ground? How high does the ladder reach on the wall?

Solution: The ladder, the wall, and the ground form a right triangle. The ladder is the hypotenuse, 8 m, and the distance on the ground is a leg, 4 m, which is half of the hypotenuse. So the triangle is semi-regular, and the angle opposite the 4 m leg, at the top of the ladder, is 30°. The angle between the ladder and the ground is then $\boxed{60°}$.

The ladder reaches as high as the longer leg: $4\sqrt{3}$ m, that is, $\boxed{4\sqrt{3} \text{ m} \approx 6.93 \text{ m}}$.

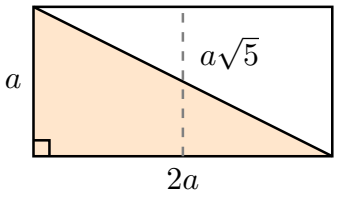
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The 1-2-$\sqrt{5}$ triangle

A domino is a rectangle made of two squares. If we cut it in half along a diagonal, we get a right triangle whose legs are a and $2a$. Because it is half of a domino, we will call it the domino triangle. (This is not standard language, just a name we use in our class. In books, you will find it as the right triangle with legs 1 and 2.) By the Pythagorean theorem, its hypotenuse c is

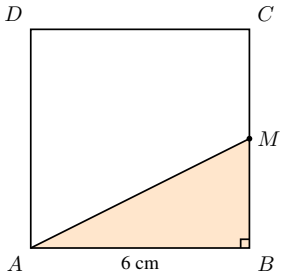
$$c^2 = a^2 + (2a)^2 = a^2 + 4a^2 = 5a^2 \qquad \text{so} \qquad c = a\sqrt{5}$$

Unlike the other famous triangles, the angles of this triangle are not nice whole numbers (they are about 26.6° and 63.4°). It is worth knowing because it hides in so many pictures: whenever a segment connects a corner of a square with the midpoint of an opposite side, this triangle appears.



The 1-2-$\sqrt{5}$ triangle: If the legs of a right triangle are a and $2a$ units long, then its hypotenuse is $a\sqrt{5}$ units long.

Example 6. The square $ABCD$ has sides 6 cm long, and M is the midpoint of the side BC. Find the exact length of the segment AM.



Solution: The triangle ABM has a right angle at B. Its legs are $AB = 6\,\text{cm}$ and $BM = 3\,\text{cm}$, so it is half of a domino, with $a = 3\,\text{cm}$. So $AM = \boxed{3\sqrt{5}\,\text{cm}}$.

Check with the Pythagorean theorem: $(3\,\text{cm})^2 + (6\,\text{cm})^2 = 9\,\text{cm}^2 + 36\,\text{cm}^2 = 45\,\text{cm}^2$, and $\sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}$.



Practice Problems

1. The legs of an isosceles right triangle are 9 cm long. Find the length of the hypotenuse.
2. The hypotenuse of an isosceles right triangle is 6 in long. Find the length of the legs.
3. The diagonal of a square is 8 ft long. Find the length of the sides and the area of the square.
4. Find the exact height and area of a regular triangle with sides 8 m long.
5. The height of a regular triangle is $6\sqrt{3}$ cm. How long are its sides?
6. The hypotenuse of a semi-regular triangle is 20 units long. Find the length of the legs.
7. The longer leg of a semi-regular triangle is 12 units long. Find the length of the shorter leg and the hypotenuse.
8. The hypotenuse of a right triangle is 14 cm long and one of its legs is 7 cm long. Find the angles of the triangle and the length of the other leg.
9. The legs of a right triangle are 5 units and 10 units long. Find the length of the hypotenuse.
10. Find the length of the diagonal of a rectangle with sides 4 ft and 8 ft.
11. A regular hexagon has sides 4 cm long. Its three longest diagonals cut it into six regular triangles. Find the exact area of the hexagon.
12. The diagonal of a rectangle makes a 30° angle with the longer side. The shorter side is 5 cm long. Find the length of the diagonal, the length of the longer side, and the area of the rectangle.

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Answers

1. $9\sqrt{2}$ cm
2. $3\sqrt{2}$ in each
3. sides: $4\sqrt{2}$ ft, area: $32\,\text{ft}^2$
4. height: $4\sqrt{3}$ m, area: $16\sqrt{3}\,\text{m}^2$
5. 12 cm
6. 10 units and $10\sqrt{3}$ units
7. shorter leg: $4\sqrt{3}$ units, hypotenuse: $8\sqrt{3}$ units
8. 30° (opposite the 7 cm leg), 60°, and 90°; the other leg is $7\sqrt{3}$ cm
9. $5\sqrt{5}$ units
10. $4\sqrt{5}$ ft
11. $24\sqrt{3}\,\text{cm}^2$
12. diagonal: 10 cm, longer side: $5\sqrt{3}$ cm, area: $25\sqrt{3}\,\text{cm}^2$

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