

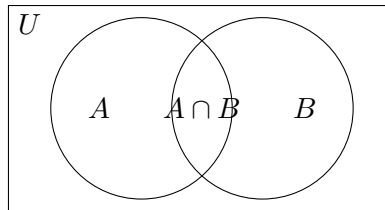
Suppose we want to count how many elements belong to the union of two sets. It is tempting to simply add the number of elements in each set, but if the sets overlap, that method counts the common elements twice. The **inclusion-exclusion principle** corrects this by first adding, then subtracting the overlap so that every element is counted exactly once.

Two Sets

Inclusion-Exclusion Principle (Two Sets). For any two finite sets A and B ,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The term $|A \cap B|$ is subtracted because those elements were counted once in $|A|$ and once again in $|B|$. Subtracting $|A \cap B|$ removes the duplicate count, leaving each element of $A \cup B$ counted exactly once.



Example 1. At a community college, 25 students in a class of 40 are taking Spanish, 18 are taking French, and 10 are taking both Spanish and French. How many students are taking Spanish or French?

Solution: Let A be the set of students taking Spanish and B be the set of students taking French. Then $|A| = 25$, $|B| = 18$, and $|A \cap B| = 10$. By the inclusion-exclusion principle,

$$|A \cup B| = 25 + 18 - 10 = 33.$$

So 33 students are taking Spanish or French (or both).

Example 2. A survey of 60 people found that 34 like coffee, 30 like tea, and 12 like both coffee and tea. How many people like neither coffee nor tea?

Solution: First we find how many people like coffee or tea.

$$|A \cup B| = 34 + 30 - 12 = 52$$

Since there are 60 people total, the number who like neither is

$$60 - 52 = 8.$$

So 8 people like neither coffee nor tea.

Example 3. How many numbers among $1, 2, 3, \dots, 100$ are divisible by 2 or 3?

Solution: Let A be the set of numbers among $1, 2, \dots, 100$ divisible by 2, and let B be the set of those divisible by 3. Every 2nd number is divisible by 2, so $|A| = \frac{100}{2} = 50$. Every 3rd number is divisible by 3, so $|B| = \left\lfloor \frac{100}{3} \right\rfloor = 33$. A number is divisible by both 2 and 3 exactly when it is divisible by 6, so $|A \cap B| = \left\lfloor \frac{100}{6} \right\rfloor = 16$. By the inclusion-exclusion principle,

$$|A \cup B| = 50 + 33 - 16 = 67.$$

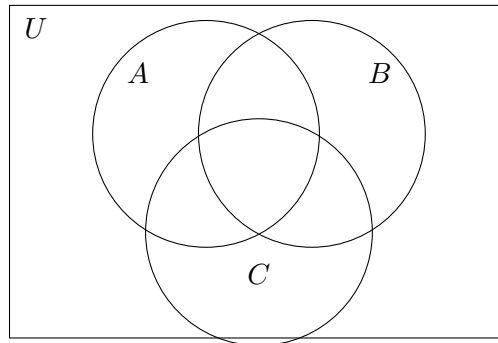
So 67 of the numbers $1, 2, \dots, 100$ are divisible by 2 or 3.

Three Sets

When we have three sets, simply adding all three sizes and subtracting each pairwise overlap once is still not enough - the elements that belong to **all three** sets get subtracted too many times and must be added back in.

Inclusion-Exclusion Principle (Three Sets). For any three finite sets A , B , and C ,

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C|. \end{aligned}$$



Example 4. Among 100 students, 45 are taking Math, 38 are taking Physics, and 21 are taking Chemistry. Of these, 18 are taking both Math and Physics, 10 are taking both Math and Chemistry, 8 are taking both Physics and Chemistry, and 5 are taking all three courses. How many of the 100 students are taking at least one of these three courses?

Solution: Let A , B , and C denote the sets of students taking Math, Physics, and Chemistry, respectively. Then

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 45 + 38 + 21 - 18 - 10 - 8 + 5 \\ &= 73. \end{aligned}$$

So 73 of the 100 students are taking at least one of Math, Physics, or Chemistry. (The remaining 27 students are taking none of the three.)

Example 5. Among the numbers $1, 2, 3, \dots, 200$, how many are divisible by 2 or 3 or 5?

Solution: Let A , B , and C be the sets of numbers among $1, 2, \dots, 200$ divisible by 2, 3, and 5, respectively. Then

$$|A| = \left\lfloor \frac{200}{2} \right\rfloor = 100, \quad |B| = \left\lfloor \frac{200}{3} \right\rfloor = 66, \quad |C| = \left\lfloor \frac{200}{5} \right\rfloor = 40.$$

A number is divisible by two of these exactly when it is divisible by the product of the two (since 2, 3, and 5 are all prime), so

$$|A \cap B| = \left\lfloor \frac{200}{6} \right\rfloor = 33, \quad |A \cap C| = \left\lfloor \frac{200}{10} \right\rfloor = 20, \quad |B \cap C| = \left\lfloor \frac{200}{15} \right\rfloor = 13,$$

and a number is divisible by all three exactly when it is divisible by $2 \cdot 3 \cdot 5 = 30$, so $|A \cap B \cap C| = \left\lfloor \frac{200}{30} \right\rfloor = 6$. By the inclusion-exclusion principle,

$$\begin{aligned} |A \cup B \cup C| &= 100 + 66 + 40 - 33 - 20 - 13 + 6 \\ &= 146. \end{aligned}$$

So 146 of the numbers $1, 2, \dots, 200$ are divisible by 2 or 3 or 5.



Practice Problems

1. At a gym, 28 of the 50 members use the treadmills, 22 use the weight machines, and 9 use both. How many members use the treadmills or the weight machines?
2. Of 80 students, 50 play basketball, 35 play soccer, and 20 play both. How many students play neither sport?
3. In a class, $|A| = 30$, $|B| = 24$, and $|A \cup B| = 45$. Find $|A \cap B|$.
4. A tour company surveyed 120 tourists. Of these, 65 visited the museum (M), 54 visited the park (P), and 40 visited the zoo (Z). Also, 25 visited both the museum and the park, 20 visited both the museum and the zoo, 18 visited both the park and the zoo, and 10 visited all three. How many tourists visited none of the three attractions?
5. Using the data from problem 4, how many tourists visited **exactly one** of the three attractions?
6. How many numbers among $1, 2, 3, \dots, 500$ are divisible by 5 or 8?



Answers

1. 41 2. 15 3. 9 4. 14 5. 63 6. 150