

Whenever money is borrowed or invested, the lender expects to be paid for letting someone else use the money. This payment is called **interest**. In this handout we look at the two basic ways interest can be calculated: **simple interest** and **compound interest**.

Definition. The original amount of money that is borrowed or invested is called the **principal**, denoted P . The **interest rate**, denoted r , is the interest paid per year as a percentage of the principal. In formulas we always write r as a decimal, so 5% means $r = 0.05$. The length of time the money is borrowed or invested is called the **term**, denoted t , and it is measured in years.

Simple Interest

With **simple interest**, the interest for each year is always calculated from the *original* principal, never from interest that has already been earned. So the account earns the same amount of interest every year, namely $P \cdot r$ dollars, and after t years the interest earned is $P \cdot r \cdot t$.

Simple Interest Formulas. If P dollars are invested at an annual simple interest rate r for t years, then the interest earned is

$$I = Prt,$$

and the accumulated amount, the principal plus the interest, is

$$A = P + I = P(1 + rt).$$

Since the same interest is added every year, the amount A grows by the same fixed number of dollars, Pr , each year. We say that simple interest produces *linear* growth.

Example 1. Suppose \$2 000 is deposited into an account paying 5% simple interest per year. Find the interest earned and the total amount in the account after four years.

Solution. Here $P = 2\,000$, $r = 0.05$, and $t = 4$.

$$I = Prt = 2\,000 \cdot 0.05 \cdot 4 = 400,$$

$$A = P + I = 2\,000 + 400 = \$2\,400.$$

Example 2. A loan of \$1 500 is repaid with a single payment of \$1 650 after two years. If simple interest was used, what annual interest rate was charged?

Solution. The interest paid is $I = 1\,650 - 1\,500 = 150$. Solving $I = Prt$ for r ,

$$r = \frac{I}{Pt} = \frac{150}{(1\,500)(2)} = 0.05 = 5\%.$$

Example 3. An account earns 5% simple interest. After four years it contains \$6 000. How much was originally deposited?

Solution. This time the principal P is the unknown. We know that $A = 6\,000$, $r = 0.05$, and $t = 4$. The amount is the principal plus the interest, $A = P + Prt$. Both terms contain P , so we factor P out:

$$6\,000 = P + P(0.05)(4) = P + 0.2P = P(1 + 0.2) = 1.2P.$$

Dividing both sides by 1.2,

$$P = \frac{6\,000}{1.2} = \$5\,000.$$

In general, solving $A = P(1 + rt)$ for P gives $P = \frac{A}{1 + rt}$. As a check, the interest on \$5 000 is $5\,000 \cdot 0.05 \cdot 4 = \$1\,000$, and $5\,000 + 1\,000 = 6\,000$.

Simple interest is common for short-term loans. When the term is given in months or days, we write t as a fraction of a year:

$$t = \frac{\text{months}}{12} \quad \text{or} \quad t = \frac{\text{days}}{360}.$$

(In this course we use a 360-day year, the *ordinary interest* convention used by many banks.)

Example 4. Find the simple interest on \$3 000 at 6% for 90 days.

Solution. The term is $t = \frac{90}{360} = 0.25$ of a year, so

$$I = Prt = 3\,000 \cdot 0.06 \cdot 0.25 = \$45.$$

Rounding. Round money amounts to the nearest cent and percentages to the nearest tenth of a percent. Keep full calculator accuracy during a calculation, and round only the final answer.

Partial Payments: The United States Rule

Sometimes a loan is repaid in several payments before the end of the term. The most common way to handle such **partial payments** with simple interest is the **United States Rule**.

The United States Rule. Interest is always simple interest on the current unpaid principal. When a partial payment is made, first compute the interest accrued since the previous payment (or since the loan began). The payment is applied to this accrued interest first, and only the remainder reduces the principal. At the end of the term, the amount owed is the unpaid principal plus the interest accrued since the last payment.

If a payment is smaller than the accrued interest, the principal does not change, and the unpaid interest simply keeps accruing; it is not added to the principal.

Example 5. A loan of \$4 000 for three years carries an annual simple interest rate of 6%. After seventeen months a partial payment of \$1 500 is made. After another eleven months a second payment of \$1 000 is made. How much is owed at the end of the three years?

Solution. The payments are made at months 17 and $17 + 11 = 28$, and the loan ends at month 36. Balances are actual amounts of money, so we round to the nearest cent after each payment.

First payment. The interest accrued in 17 months is

$$I = 4\,000 \cdot 0.06 \cdot \frac{17}{12} = \$340.$$

Of the \$1 500, \$340 pays the interest, and the remaining $1\,500 - 340 = \$1\,160$ reduces the principal, which becomes $4\,000 - 1\,160 = \$2\,840$.

Second payment. The interest accrued in the next 11 months is

$$I = 2\,840 \cdot 0.06 \cdot \frac{11}{12} = \$156.20.$$

So $1\,000 - 156.20 = \$843.80$ reduces the principal, which becomes $2\,840 - 843.80 = \$1\,996.20$.

End of the loan. The interest accrued in the last $36 - 28 = 8$ months is

$$I = 1\,996.20 \cdot 0.06 \cdot \frac{8}{12} \approx \$79.85.$$

The amount owed is $1\,996.20 + 79.85 = \$2\,076.05$.

Compound Interest

In most savings accounts and long-term investments, the interest that has been earned is added to the principal, and from then on it earns interest too. This is called **compound interest**, informally “interest on interest.” In this handout the interest is added once a year, and we say that the interest is **compounded annually**.

Suppose P dollars are invested at an annual rate r , compounded annually. After one year the balance is

$$A_1 = P + Pr = P(1 + r).$$

In the second year, interest is earned on A_1 , not just on P :

$$A_2 = A_1 + A_1 r = A_1(1 + r) = P(1 + r)^2.$$

Each year the balance is multiplied by $1 + r$, so after t years we have the following formula.

Compound Interest Formula (compounded annually). If P dollars are invested at an annual rate r , compounded annually, then the accumulated amount after t years is

$$A = P(1 + r)^t.$$

The interest earned is $I = A - P$.

Example 6. \$2 000 is invested at 5% compounded annually for four years. Find the accumulated amount and the interest earned, and compare with the simple interest account.

Solution. Here $P = 2\,000$, $r = 0.05$, and $t = 4$.

$$A = 2\,000(1 + 0.05)^4 = 2\,000(1.05)^4 \approx \$2\,431.01.$$

The interest earned is $I = 2431.01 - 2\,000 = \$431.01$. With simple interest at the same rate the interest would have been only \$400. The table shows where the extra \$31.01 comes from: each year the compound account earns interest on a larger balance.

Year	Simple interest	Compound interest
0	\$2 000.00	\$2 000.00
1	\$2 100.00	\$2 100.00
2	\$2 200.00	\$2 205.00
3	\$2 300.00	\$2 315.25
4	\$2 400.00	\$2 431.01

Example 7. Find the balance after 6 years if \$8 000 is deposited into an account that pays 3% compounded annually.

Solution. Here $P = 8\,000$, $r = 0.03$, and $t = 6$.

$$A = 8\,000(1.03)^6 \approx \$9\,552.42.$$

Comparing Simple and Compound Growth

Simple interest grows *linearly*: the same dollar amount is added every year. Compound interest grows *exponentially*: the balance is multiplied by the same factor every year, so the amount added each year keeps getting larger. For short terms the two are close, but for long terms compound interest wins by a wide margin. The table shows \$1 000 invested at 5%.

Years	Simple interest $P(1 + rt)$	Compound interest $P(1 + r)^t$
0	\$1 000.00	\$1 000.00
5	\$1 250.00	\$1 276.28
10	\$1 500.00	\$1 628.89
20	\$2 000.00	\$2 653.30
30	\$2 500.00	\$4 321.94
40	\$3 000.00	\$7 039.99

After 40 years the compound interest account holds more than twice as much as the simple interest account.

Example 8. Maria invests \$5 000 for 40 years at 6%.

- How much will she have if the interest is simple interest?
- How much will she have if the interest is compounded annually?
- How much interest does each account earn during the 40th year alone?

Solution.

- $A = 5\,000(1 + 0.06 \cdot 40) = 5\,000(3.4) = \$17\,000$.
- $A = 5\,000(1.06)^{40} \approx \$51\,428.59$. The compound account holds about three times as much as the simple account. It has earned $\$51\,428.59 - \$5\,000 = \$46\,428.59$ in interest, compared with \$12 000.
- With simple interest, every year brings the same interest, $5\,000 \cdot 0.06 = \$300$. With compound interest, the balance after 39 years is $5\,000(1.06)^{39} \approx \$48\,517.54$, so the interest earned in the 40th year is $0.06 \cdot \$48\,517.54 \approx \$2\,911.05$. In that single year, the compound account earns almost ten times as much as the simple account.

The longer the money stays in the account, the larger the balance that earns interest, and this is why compound interest becomes so much stronger in the long run.

So far the interest was added once a year. Banks often add interest more often, for example every month or every day. That changes the answer slightly, and it is the topic of the next handout, *The Compound Interest Formula*.



Practice Problems

Round money to the nearest cent and percentages to the nearest tenth of a percent.

1. Find the simple interest earned on a principal of \$3 500 invested at 4.5% for six years.
2. A \$5 000 loan is repaid with \$5 900 after four years. If simple interest was used, find the annual interest rate.
3. Find the simple interest on \$2 500 at 3.6% for 150 days. (Use a 360-day year.)
4. How many years does it take for \$2 000 to earn \$600 in interest at 5% simple interest?
5. A savings account earns 4% simple interest. After six years it holds \$4 960. How much was originally deposited?
6. A \$6 000 loan for two years carries 9% simple interest. After nine months a partial payment of \$2 000 is made, and after 8 more months a second payment of \$1 500 is made. Use the United States Rule to find the amount owed at the end of the two years.
7. Find the accumulated value of \$4 000 invested at 6% compounded annually for eight years.
8. \$1 500 is invested at 4% for ten years. Find the accumulated amount and the interest earned if the interest is
 - (a) simple interest,
 - (b) compounded annually.
9. An account earns simple interest. If the term is doubled, what happens to the interest earned? Does the total amount in the account also double? Explain.



Answers

1. \$945.00
2. $\frac{900}{5\,000 \cdot 4} = 0.045$, so 4.5%.
3. $t = \frac{150}{360}$, so $I = 2\,500 \cdot 0.036 \cdot \frac{150}{360} = \37.50 .
4. $t = \frac{I}{Pr} = \frac{600}{2\,000 \cdot 0.05} = 6$ years.
5. $4\,960 = P + P(0.04)(6) = P(1 + 0.24) = 1.24P$, so $P = \frac{4\,960}{1.24} = \$4\,000$.
6. First payment (month 9): interest $6\,000 \cdot 0.09 \cdot \frac{9}{12} = \405 , so \$1 595 reduces the principal to \$4 405. Second payment (month 17): interest $4\,405 \cdot 0.09 \cdot \frac{8}{12} = \264.30 , so \$1 235.70 reduces the principal to \$3 169.30. Last 7 months: interest $3\,169.30 \cdot 0.09 \cdot \frac{7}{12} \approx \166.39 . The amount owed is $3\,169.30 + 166.39 = \$3\,335.69$.
7. $A = 4\,000(1.06)^8 \approx \$6\,375.39$
8. a) $A = 1\,500(1 + 0.04 \cdot 10) = \$2\,100.00$, and the interest is \$600. b) $A = 1\,500(1.04)^{10} \approx \$2\,220.37$, and the interest is \$720.37.
9. The interest $I = Prt$ doubles, because it is proportional to t . The total amount does not double, because the principal does not change: for example, \$1 000 at 5% gives \$1 250 after 5 years but \$1 500 after 10 years.