Logic studies how the truth of some sentences depends on the truth of others. The sentences it takes as its raw material are *statements*: declarative sentences that are either true or false, but not both.

Definition. A **statement** (or **proposition**) is a declarative sentence that has exactly one **truth value**, either **true** (T) or **false** (F).

"The number 7 is prime" is a statement (a true one). "The number 8 is prime" is also a statement (a false one). "Close the door" and "What time is it?" are not statements, because they are not declarative.

A statement that contains no other statement as a part is called **simple** (or **atomic**). We abbreviate simple statements by capital letters $P, Q, R, \ldots$, called **statement variables**. A statement built from simpler statements by means of words such as *and*, *or*, and *if–then* is called **compound**.

Definition. A **logical connective** is a word or phrase that combines one or more statements into a new statement whose truth value is completely determined by the truth values of the parts.
This section treats the three connectives that join two statements:

Name	English	Symbol	Read as
Conjunction	*and*	$P \wedge Q$	"P and Q"
Disjunction	*or*	$P \vee Q$	"P or Q"
Conditional	*if–then*	$P \to Q$	"if P, then Q"

The tool we use to record the meaning of a connective is the **truth table**. It lists every possible combination of truth values for the simple parts and, in the final column, the resulting truth value of the compound statement. Two simple statements give $2 \times 2 = 4$ rows.

Conjunction: *and*

Definition. The **conjunction** of statements P and Q, written $P \wedge Q$, is the statement "P and Q." It is true exactly when *both* P and Q are true, and false otherwise.

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

A single false conjunct is enough to make the whole conjunction false.

Example 1 Let P be "12 is even" (true) and Q be "12 is a multiple of 3" (true). Then $P \wedge Q$, "12 is even and 12 is a multiple of 3," is true. If instead Q were "12 is a multiple of 5" (false), the conjunction would be false, even though P is still true.

Disjunction: *or*

Definition: The **disjunction** of statements P and Q, written $P \vee Q$, is the statement "P or Q." It is true when *at least one* of P, Q is true, and false only when both are false.

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

Inclusive versus exclusive *or*. The connective $\vee$ is the **inclusive** "or": it allows the case in which both parts are true (row 1 above). English *or* is usually inclusive — "The applicant must know French or German" certainly does not disqualify someone who knows both. Occasionally English uses the **exclusive** "or," meaning "one or the other but not both," as in "The set menu comes with soup or salad." Unless a sentence clearly rules out the "both" case, translate *or* as the inclusive $\vee$.

Example 2 Let A be "$x > 0$" and B be "$x < 10$." For $x = 5$ both are true, so A or B is true. For $x = 20$, A is true and B is false, so A or B is still true. In fact A or B is false for no real number x at all, since every x satisfies at least one of the two inequalities.

The Conditional: *if – then*

Definition: The **conditional** with **antecedent** P and **consequent** Q, written $P \to Q$, is the statement "if P, then Q." It is false in the single case where P is true and Q is false, and true in every other case.

P	Q	if P, then Q
T	T	T
T	F	F
F	T	T
F	F	T

Why the last two rows are true. A conditional is a *promise* about what holds when the antecedent is met. Consider the pledge

"If you finish the assignment, then you may go out."

The promise is broken only if you finish the assignment and are *still* kept in — that is, antecedent true, consequent false (row 2). If you do not finish the assignment (rows 3 and 4), the promise has not been broken, whatever happens next, so we count it as true. A conditional whose antecedent is false is said to be **vacuously true**. Thus "If $2 + 2 = 5$, then the moon is made of cheese" is a true statement.

Ways of saying *if – then*. The following English phrases all translate to $P \to Q$:

English	Symbolization
If P, then Q.	$P \to Q$
Q if P.	$P \to Q$
P implies Q.	$P \to Q$
P is sufficient for Q.	$P \to Q$
Q is necessary for P.	$P \to Q$
Q whenever P.	$P \to Q$

The biconditional. When both $P \to Q$ and its converse $Q \to P$ hold, we write $P \leftrightarrow Q$, read "P if and only if Q" (often abbreviated "P iff Q"). It is true exactly when P and Q have the same truth value.

Summary

P	Q	P and Q	P or Q	If P, then Q
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

- **Conjunction** P and Q: true only when both parts are true.
- **Disjunction** P or Q: false only when both parts are false (inclusive "or").
- **Conditional** If P, then Q: false only when the antecedent is true and the consequent is false; vacuously true whenever the antecedent is false.

Negation: *not*

The three connectives above join two statements. Negation is different: it acts on a *single* statement and reverses its truth value.

Definition: The **negation** of a statement P, written $\neg P$, is the statement "not P," or "it is not the case that P." It is true when P is false, and false when P is true.

P	not P
T	F
F	T

A single simple statement has only two possible truth values, so the table has only 2 rows. A statement and its negation can never be true at the same time, and they can never be false at the same time: exactly one of them is true.

Example 3 Let P be "7 is a prime number" (true). Then $\neg P$ is "7 is not a prime number," which is false. Let Q be "$2+2=5$" (false). Then $\neg Q$ is "$2+2 \neq 5$," which is true.

Example 4 For a given number x, the negation of "$x > 3$" is "$x \leq 3$," and *not* "$x < 3$." The number $x = 3$ makes both "$x > 3$" and "$x < 3$" false, but a statement and its negation cannot both be false. Every number that does not satisfy $x > 3$ must satisfy the negation, and that includes $x = 3$.

Double negation. Negating twice brings back the original truth value. "It is not the case that 7 is not a prime number" says the same as "7 is a prime number."

Negating quantified statements

Statements containing words such as *all*, *every*, *some*, *there is*, or *no* are called **quantified statements**. They need extra care: the negation of "all" is *not* "none."

Example 5 Let P be "Every student in this room has a calculator." Its negation is "Not every student in this room has a calculator," that is, "*At least one* student in this room does not have a calculator."

The sentence "No student in this room has a calculator" is *not* the negation of P. If half the class has a calculator and half does not, then P is false and "No student has a calculator" is also false. A statement and its negation cannot both be false.

Statement	Negation	
All A are B.	Some A is not B.	(At least one A is not B.)
Some A is B.	No A is B.	
No A is B.	Some A is B.	
Some A is not B.	All A are B.	

In mathematics, "some" means "at least one."

Example 6 Each statement is followed by its negation. In each pair, exactly one of the two is true.

(a) "All prime numbers are odd." This is false, because 2 is prime and even.

Negation: "There is a prime number that is not odd." This is true.

(b) "There is a real number x such that $x^2 = -1$." This is false, because the square of a real number is never negative.

Negation: "For every real number x, $x^2 \neq -1$." This is true.

(c) "No perfect square ends in the digit 7." This is true: perfect squares end in 0, 1, 4, 5, 6, or 9.

Negation: "Some perfect square ends in the digit 7." This is false.

(d) "Some triangles have two obtuse angles." This is false, because two obtuse angles alone add up to more than 180°.

Negation: "No triangle has two obtuse angles." This is true.



Practice Problems

1. Classify each sentence as a statement or not. If it is a statement, give its truth value.
   (a) 5 is an odd number.
   (b) Is 5 an odd number?
   (c) $2 + 1 = 4$.
   (d) Every even number greater than 2 is a sum of two primes.
2. Determine about each statement whether they are true or false.
   (a) 2 is a prime number and 5 is even.
   (b) 2 is a prime number or 5 is even.
   (c) If 2 is a prime number, then 5 is even.
   (d) If 5 is even, then 2 is a prime number.
   (e) $-3 \in \mathbb{Z}$ and the angles of a regular triangle are all 60°.
   (f) $-3 \in \mathbb{Z}$ or the angles of a regular triangle are all 60°.
   (g) If $-3 \in \mathbb{Z}$, then the angles of a regular triangle are all 60°.
   (h) If the angles of a regular triangle are all 60°, then $-3 \in \mathbb{Z}$.
   (i) 7 is a triangular number and 8 is a prime.
   (j) 7 is a triangular number or 8 is a prime.
   (k) If 7 is a triangular number, then 8 is a prime.
   (l) If 8 is a prime, then 7 is a triangular number.
3. Translate back into idiomatic English if P means "Sue is in a good mood", Q means "today is Friday", and R means "we are going out tonight."
   (a) $(P \wedge Q) \rightarrow R$
   (b) $P \rightarrow (Q \vee R)$
4. Determine whether the given statements are true or false.
   (a) If a number is divisible by 2, then it is also divisible by 4.

(b) If a number is divisible by 6, then it is also divisible by 2.

(c) If a and b are integers and ab is divisible by 10, then a or b is divisible by 10.

(d) If a and b are integers and ab is divisible by 7, then a or b is divisible by 7.

5. Construct a full truth table for each formula.

(a) $P \to (Q \text{ or } R)$

(b) $(P \text{ and } Q) \to P$

(c) $(P \to Q) \text{ and } (Q \to P)$

6. Write the negation of each statement. Then decide which one is true: the statement or its negation.

(a) 9 is an odd number.

(b) 15 is divisible by 4.

(c) All squares are rectangles.

(d) Some rectangles are squares.

(e) No even number is prime.

(f) There is a largest integer.

(g) Every real number has a square root that is a real number.

7. For a given number x, write the negation of each statement.

(a) $x < -2$

(b) $x \geq 10$



Answers

1. a) true statement b) not a statement c) false statement

   d) statement, not yet proven or dis-proven (called the Goldbach conjecture)

2. a) false b) true c) false d) true e) true f) true g) true

   h) true i) false j) false k) true l) true

3. a) If Sue is in a good mood, and today is Friday, then we are going out tonight.

   b) If Sue is in a good mood, then it is Friday or we are going out tonight.

4. a) false b) true c) false d) true

5. a)

P	Q	R	If P, then Q or R
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

   b)

P	Q	If P and Q, then P
T	T	T
T	F	T
F	T	T
F	F	T

   c)

P	Q	If P, then Q, and if Q, then P
T	T	T
T	F	F
F	T	F
F	F	T

6. a) 9 is not an odd number. The statement is true.

   b) 15 is not divisible by 4. The negation is true.

   c) Some square is not a rectangle. The statement is true.

   d) No rectangle is a square. The statement is true.

   e) Some even number is prime. The negation is true, because 2 is even and prime.

   f) There is no largest integer, that is, every integer has a larger integer. The negation is true.

   g) Some real number does not have a real square root. The negation is true, for example -1.

7. a) $x \geq -2$ b) $x < 10$