

This handout collects three more topics about linear equations: solving equations with the unknown on both sides, classifying equations as conditional, identities, or contradictions, and solving equations in which we must first multiply algebraic expressions together.

Part 1 – The Unknown on Both Sides

When the unknown appears on both sides of the equation, we first move all the unknown terms to one side. We can move them to either side, but it is often more convenient to move them so that the coefficient of the unknown ends up positive.

Example 1. Solve each of the given equations. Make sure to check your solutions.

a) $2x - 5 = 7x + 10$

b) $3(2x - 1) - 5(2x - 3) = 4x - 4$

Solution: a) We move the unknown to one side and the numbers to the other.

$$\begin{aligned} 2x - 5 &= 7x + 10 && \text{subtract } 2x \\ -5 &= 5x + 10 && \text{subtract } 10 \\ -15 &= 5x && \text{divide by } 5 \\ -3 &= x \end{aligned}$$

We check: if $x = -3$, then

$$\text{LHS} = 2(-3) - 5 = -6 - 5 = -11 \quad \text{and} \quad \text{RHS} = 7(-3) + 10 = -21 + 10 = -11$$

So our solution, $\boxed{x = -3}$ is correct.

b) Here we must first apply the distributive law before we can move the unknown to one side.

$$\begin{aligned} 3(2x - 1) - 5(2x - 3) &= 4x - 4 && \text{distribute } 3 \text{ and } -5 \\ 6x - 3 - 10x + 15 &= 4x - 4 && \text{combine like terms} \\ -4x + 12 &= 4x - 4 && \text{add } 4x \\ 12 &= 8x - 4 && \text{add } 4 \\ 16 &= 8x && \text{divide by } 8 \\ 2 &= x \end{aligned}$$

We check: if $x = 2$, then

$$\text{LHS} = 3(2 \cdot 2 - 1) - 5(2 \cdot 2 - 3) = 3(3) - 5(1) = 9 - 5 = 4$$

$$\text{RHS} = 4(2) - 4 = 8 - 4 = 4$$

So our solution, $\boxed{x = 2}$ is correct.

As before, we can use this idea to solve application problems in which two quantities – often two costs – are being compared.

Example 2. A car rental company charges \$80 plus \$25 per day. Another company charges \$140 plus \$15 per day. After how many days would the two companies charge the same total amount?

Solution: Let us denote the number of days by x . The first company's total charge is $80 + 25x$, and the second company's total charge is $140 + 15x$. We are looking for the number of days for which these two amounts are equal.

$$\begin{aligned} 80 + 25x &= 140 + 15x && \text{subtract } 15x \\ 80 + 10x &= 140 && \text{subtract } 80 \\ 10x &= 60 && \text{divide by } 10 \\ x &= 6 \end{aligned}$$

We check: after 6 days, the first company charges $80 + 25(6) = 80 + 150 = 230$ dollars, and the second company charges $140 + 15(6) = 140 + 90 = 230$ dollars. The two amounts agree, so after 6 days the two companies charge the same total amount.

Example 3. Gym A charges \$10 a month, plus a one-time \$150 enrollment fee. Gym B charges \$25 a month, but is running a promotion: the first 3 months are free, so over x months, a member only pays for $(x - 3)$ of them. After how many months would the total amount paid to each gym be the same?

Solution: Let us denote the number of months by x . The total amount paid to Gym A is $10x + 150$. The total amount paid to Gym B is $25(x - 3)$, the monthly rate times the number of months actually charged. We set the two totals equal and solve.

$$\begin{aligned} 10x + 150 &= 25(x - 3) && \text{distribute } 25 \\ 10x + 150 &= 25x - 75 && \text{subtract } 10x \\ 150 &= 15x - 75 && \text{add } 75 \\ 225 &= 15x && \text{divide by } 15 \\ 15 &= x \end{aligned}$$

We check: over 15 months, Gym A charges $10(15) + 150 = 300$ dollars. Gym B charges the member for $15 - 3 = 12$ months, so its total is $25(12) = 300$ dollars. The two totals agree, so after 15 months the two gyms would have charged the same total amount.

Part 2 – Conditional Equations, Identities, and Contradictions

Once the unknown appears on both sides, we might face new situations that were not possible before.

Example 4. Solve each of the given equations.

$$\text{a) } 5y + 16 = 5y + 16 \qquad \text{b) } -6x + 5 = -6x + 9$$

Solution: a) This equation looks different from the others because the two sides are identical.

$$\begin{aligned} 5y + 16 &= 5y + 16 && \text{subtract } 5y \\ 16 &= 16 \end{aligned}$$

The statement $16 = 16$ is true no matter what the value of y is. Such a statement is called an **unconditionally true statement**, or an **identity**. All numbers are solutions of this equation.

b)

$$\begin{aligned} -6x + 5 &= -6x + 9 \quad \text{add } 6x \\ 5 &= 9 \end{aligned}$$

When we tried to eliminate the unknown from one side, it disappeared from both sides. We are left with the statement $5 = 9$, which is false no matter what the value of x is. Such a statement is called an **unconditionally false statement**, or a **contradiction**, and this equation **has no solution**.

These strange situations happen exactly when the unknown has the same coefficient on both sides. Based on their solution sets, linear equations can be classified as follows.

1. If the last line is of the form $x = 5$, the equation is called **conditional**. This is because the truth value of the statement depends on the value of x : true if x is 5, false otherwise. A conditional equation has exactly **one solution**.
2. If the last line is of the form $1 = 1$, the equation is unconditionally true. Such an equation is called an **identity**, and **all real numbers are solutions** of it.
3. If the last line is of the form $3 = 8$, the equation is unconditionally false. Such an equation is called a **contradiction**, and its solution set is **the empty set**.



Discussion: Classify each of the given equations as a conditional equation, an identity, or a contradiction.

- | | |
|----------------------|----------------------|
| a) $3x + 1 = 3x - 1$ | b) $2x - 4 = 7x - 4$ |
| c) $x - 4 = 4 - x$ | d) $x - 1 = -1 + x$ |

Part 3 – Multiplying Expressions Together

Sometimes an equation contains a product of two algebraic expressions on each side. At first, such an equation looks quadratic because of the x^2 terms produced when we multiply. But if the coefficient of x^2 turns out to be the same on both sides, those terms cancel, and we are left with an ordinary linear equation.

Example 5. Solve the given equation. Make sure to check your solution.

$$(2x - 3)^2 - (x + 1)(3x - 5) = 11 - (x - 1)(3 - x)$$

Solution: We carefully expand the indicated products and combine like terms. Notice that even after we expand $(x + 1)(3x - 5)$ and $(x - 1)(3 - x)$, we still need to keep them in parentheses because we are subtracting them. We will first work out the products.

$$\begin{aligned} (2x - 3)^2 &= (2x - 3)(2x - 3) = 4x^2 - 6x - 6x + 9 = 4x^2 - 12x + 9 \\ (x + 1)(3x - 5) &= 3x^2 - 5x + 3x - 5 = 3x^2 - 2x - 5 \\ (x - 1)(3 - x) &= 3x - x^2 - 3 + x = -x^2 + 4x - 3 \end{aligned}$$

We are now ready to begin solving the equation.

$$\begin{aligned}
 (2x - 3)^2 - (x + 1)(3x - 5) &= 11 - (x - 1)(3 - x) \\
 4x^2 - 12x + 9 - (3x^2 - 2x - 5) &= 11 - (-x^2 + 4x - 3) && \text{to subtract is to add the opposite} \\
 4x^2 - 12x + 9 - 3x^2 + 2x + 5 &= 11 + x^2 - 4x + 3 && \text{combine like terms} \\
 x^2 - 10x + 14 &= x^2 - 4x + 14 && \text{subtract } x^2 \\
 -10x + 14 &= -4x + 14 && \text{add } 10x \\
 14 &= 6x + 14 && \text{subtract } 14 \\
 0 &= 6x && \text{divide by } 6 \\
 0 &= x
 \end{aligned}$$

We check: if $x = 0$, then

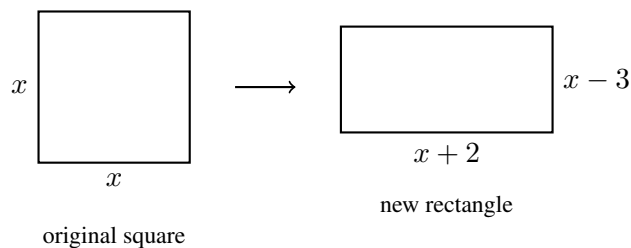
$$\begin{aligned}
 \text{LHS} &= (2 \cdot 0 - 3)^2 - (0 + 1)(3 \cdot 0 - 5) = (-3)^2 - 1(-5) = 9 + 5 = 14 \\
 \text{RHS} &= 11 - (0 - 1)(3 - 0) = 11 - (-1)3 = 11 + 3 = 14
 \end{aligned}$$

and so our solution, $x = 0$ is correct.

This same idea – multiplying expressions together and watching the quadratic terms cancel – also shows up in application problems.

Example 6. If we increase one side of a square by 2 units and decrease the other side by 3 units, the area decreases by 16 square units. How long are the sides of the square?

Solution: Let us denote the side of the square by x . Then its area is x^2 . If we increase one side by 2 units and decrease the other side by 3 units, the new figure is a rectangle with sides $x + 2$ and $x - 3$, and its area is $(x + 2)(x - 3)$. The equation will express the fact that the area decreased by 16 square units.



$$\begin{aligned}
 x^2 - (x + 2)(x - 3) &= 16 && \text{multiply out } (x + 2)(x - 3) \\
 x^2 - (x^2 - x - 6) &= 16 && \text{to subtract is to add the opposite} \\
 x^2 - x^2 + x + 6 &= 16 && \text{combine like terms} \\
 x + 6 &= 16 && \text{subtract } 6 \\
 x &= 10
 \end{aligned}$$

Thus the square has sides 10 units long. We check: the area of the original square is $10^2 = 100$ square units. The new rectangle has sides 12 and 7, so its area is $12 \cdot 7 = 84$ square units. Indeed, $100 - 84 = 16$, so our solution is correct.

Example 7. There are x people in a room. They all shake hands with every other person in the room (but not with themselves). If there were 6 more people in the room, the number of handshakes would increase by 75. How many people are in the room?

Solution: If there are n people in a room and everybody shakes hands with everybody else exactly once, the number of handshakes is $\frac{n(n-1)}{2}$: each of the n people shakes hands with the other $n-1$ people, and this counts every handshake twice (once from each participant's point of view), hence the division by 2.

So if there are x people in the room, the number of handshakes is $\frac{x(x-1)}{2}$. If there were 6 more people, that is, $x+6$ people, the number of handshakes would be $\frac{(x+6)(x+5)}{2}$. The equation expresses that this second amount is 75 more than the first.

$$\begin{aligned}\frac{(x+6)(x+5)}{2} &= \frac{x(x-1)}{2} + 75 && \text{multiply both sides by 2} \\ (x+6)(x+5) &= x(x-1) + 150 && \text{multiply out both products} \\ x^2 + 11x + 30 &= x^2 - x + 150 && \text{subtract } x^2 \\ 11x + 30 &= -x + 150 && \text{add } x \\ 12x + 30 &= 150 && \text{subtract 30} \\ 12x &= 120 && \text{divide by 12} \\ x &= 10\end{aligned}$$

So there are 10 people in the room. We check: with 10 people, the number of handshakes is $\frac{10 \cdot 9}{2} = 45$. With 6 more people, that is 16 people, the number of handshakes is $\frac{16 \cdot 15}{2} = 120$. Indeed, $120 - 45 = 75$, so our solution is correct.



Sample Problems

Solve each of the following equations. If the equation is an identity or a contradiction, state that instead of a numerical answer.

1. $4x - 9 = x + 12$

3. $5(x+2) - 2(x-3) = 3(x+4) + 4$

2. $3(x-4) - 2(2x+1) = -x-9$

4. $(x+4)^2 - (x-2)(x+6) = 40$

Solve each of the following application problems.

5. An internet company charges \$45 a month, plus a \$60 one-time installation fee. Another company charges \$65 a month with no installation fee. After how many months would the total cost of both plans be the same?
6. If we increase one side of a square by 4 units and decrease the other side by 1 unit, the area increases by 8 square units. How long are the sides of the square?
7. There are x people at a party. They all shake hands with every other person at the party (but not with themselves). If there were 4 more people at the party, the number of handshakes would increase by 54. How many people are at the party?



Practice Problems

Solve each of the following equations. If the equation is an identity or a contradiction, state that instead of a numerical answer.

1. $6x - 11 = 2x + 9$

5. $5(x + 2) - 3(x - 4) = 2x + 9$

2. $-3x + 8 = 5x - 16$

6. $8 - 3(x - 5) = 3(4 - x) + x$

3. $3(2x - 5) - 4(x - 1) = -x + 7$

7. $(x + 5)^2 - (x - 2)(x + 9) = 58$

4. $4(x - 3) - 2(x + 1) = 2x - 14$

8. $(x - 3)^2 - (x + 1)(x - 5) = 4$

Solve each of the following application problems.

9. A moving company charges \$250 plus \$35 per hour. Another company charges \$100 plus \$60 per hour. For how many hours of work would the two companies charge the same total amount?
10. If we increase one side of a square by 5 units and decrease the other side by 2 units, the area increases by 14 square units. How long are the sides of the square?
11. There are x people at a party. They all shake hands with every other person at the party (but not with themselves). If there were 4 more people at the party, the number of handshakes would increase by 54. How many people are at the party?



Answers

Sample Problems

1. $x = 7$ 2. contradiction, no solution 3. identity, all real numbers are solutions 4. $x = 3$
5. After 3 months. 6. The sides are 4 units long. 7. There are 12 people at the party.

Practice Problems

1. $x = 5$ 2. $x = 3$ 3. $x = 6$ 4. identity, all real numbers are solutions 5. contradiction, no solution
6. $x = 11$ 7. $x = 5$ 8. $x = 5$
9. After 6 hours. 10. The sides are 8 units long. 11. There are 12 people at the party.

Sample Problems Solutions

Solve each of the following equations. If the equation is an identity or a contradiction, state that instead of a numerical answer.

1. $4x - 9 = x + 12$

Solution:

$$4x - 9 = x + 12 \quad \text{subtract } x$$

$$3x - 9 = 12 \quad \text{add 9}$$

$$3x = 21 \quad \text{divide by 3}$$

$$x = 7$$

We check: if $x = 7$, then

$$\text{LHS} = 4(7) - 9 = 28 - 9 = 19 = \text{RHS} = 7 + 12 = 19 \checkmark$$

Thus our solution, $x = 7$, is correct.

2. $3(x - 4) - 2(2x + 1) = -x - 9$

Solution: We distribute and combine like terms.

$$3(x - 4) - 2(2x + 1) = -x - 9 \quad \text{distribute 3 and } -2$$

$$3x - 12 - 4x - 2 = -x - 9 \quad \text{combine like terms}$$

$$-x - 14 = -x - 9 \quad \text{add } x$$

$$-14 = -9$$

The last line is a false statement, no matter what the value of x is. Thus this equation is a contradiction, and it has no solution.

3. $5(x + 2) - 2(x - 3) = 3(x + 4) + 4$

Solution: We distribute and combine like terms on both sides.

$$5(x + 2) - 2(x - 3) = 3(x + 4) + 4 \quad \text{distribute}$$

$$5x + 10 - 2x + 6 = 3x + 12 + 4 \quad \text{combine like terms}$$

$$3x + 16 = 3x + 16$$

The last line is true no matter what the value of x is. Thus this equation is an identity, and all real numbers are solutions.

4. $(x + 4)^2 - (x - 2)(x + 6) = 40$

Solution: We first work out the two products.

$$(x + 4)^2 = x^2 + 8x + 16$$

$$(x - 2)(x + 6) = x^2 + 4x - 12$$

We substitute these back into the equation.

$$x^2 + 8x + 16 - (x^2 + 4x - 12) = 40 \quad \text{to subtract is to add the opposite}$$

$$x^2 + 8x + 16 - x^2 - 4x + 12 = 40 \quad \text{combine like terms}$$

$$4x + 28 = 40 \quad \text{subtract 28}$$

$$4x = 12 \quad \text{divide by 4}$$

$$x = 3$$

We check: if $x = 3$, then $\text{LHS} = (7)^2 - (1)(9) = 49 - 9 = 40 = \text{RHS} \checkmark$ Thus our solution, $x = 3$, is correct.

5. An internet company charges \$45 a month, plus a \$60 one-time installation fee. Another company charges \$65 a month with no installation fee. After how many months would the total cost of both plans be the same?

Solution: Let us denote the number of months by x . The first company's total cost is $45x + 60$, and the second company's total cost is $65x$.

$$45x + 60 = 65x \quad \text{subtract } 45x$$

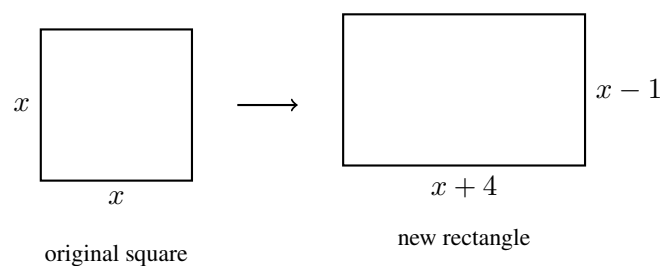
$$60 = 20x \quad \text{divide by 20}$$

$$3 = x$$

We check: after 3 months, the first company's total is $45(3) + 60 = 135 + 60 = 195$, and the second company's total is $65(3) = 195$. The totals agree, so after 3 months the two plans cost the same.

6. If we increase one side of a square by 4 units and decrease the other side by 1 unit, the area increases by 8 square units. How long are the sides of the square?

Solution: Let us denote the side of the square by x . The new rectangle has sides $x + 4$ and $x - 1$.



$$(x + 4)(x - 1) - x^2 = 8 \quad \text{multiply out } (x + 4)(x - 1)$$

$$x^2 + 3x - 4 - x^2 = 8 \quad \text{combine like terms}$$

$$3x - 4 = 8 \quad \text{add 4}$$

$$3x = 12 \quad \text{divide by 3}$$

$$x = 4$$

Thus the square has sides 4 units long. We check: the original area is 16 square units, and the new rectangle has sides 8 and 3, so its area is 24 square units. Indeed, $24 - 16 = 8$, so our solution is correct.

7. There are x people at a party. They all shake hands with every other person at the party (but not with themselves). If there were 4 more people at the party, the number of handshakes would increase by 54. How many people are at the party?

Solution: With x people at the party, the number of handshakes is $\frac{x(x-1)}{2}$. With 4 more people, that is $x+4$ people, the number of handshakes would be $\frac{(x+4)(x+3)}{2}$. The equation expresses that this second amount is 54 more than the first.

$$\begin{aligned}\frac{(x+4)(x+3)}{2} &= \frac{x(x-1)}{2} + 54 && \text{multiply both sides by 2} \\ (x+4)(x+3) &= x(x-1) + 108 && \text{multiply out both products} \\ x^2 + 7x + 12 &= x^2 - x + 108 && \text{subtract } x^2 \\ 7x + 12 &= -x + 108 && \text{add } x \\ 8x + 12 &= 108 && \text{subtract 12} \\ 8x &= 96 && \text{divide by 8} \\ x &= 12\end{aligned}$$

So there are 12 people at the party. We check: with 12 people, the number of handshakes is $\frac{12 \cdot 11}{2} = 66$. With 4 more people, that is 16 people, the number of handshakes is $\frac{16 \cdot 15}{2} = 120$. Indeed, $120 - 66 = 54$, so our solution is correct.