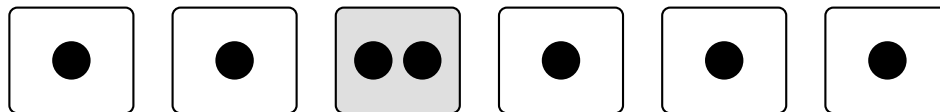


Suppose 10 pigeons fly into 9 holes. We cannot know which pigeon chooses which hole, but we can say one thing for sure: at least one hole will contain two or more pigeons. This idea is so obvious that it hardly seems worth stating – yet it is a surprisingly powerful tool for showing that something *must* happen.

The Pigeonhole Principle. If $n + 1$ or more objects are placed into n boxes, then at least one box contains at least two objects.

Why it is true. Suppose, to the contrary, that every box contained at most one object. Then the total number of objects would be at most n . But there are $n + 1$ or more objects, a contradiction.



7 pigeons in 6 holes: some hole must hold at least two.

In applications, the objects and boxes are rarely pigeons and holes. The real work is to decide *what the boxes are*. We follow three steps: decide what the objects are, decide what the boxes are, and check that there are more objects than boxes.

Example 1. Prove that among any 13 people, at least two were born in the same month.

Proof. The objects are the 13 people. The boxes are the 12 months of the year: we place each person in the box of the month in which they were born. There are more objects than boxes ($13 > 12$), so some box contains at least two people. These two people were born in the same month. ■

Example 2. A drawer contains many black socks and many blue socks, all mixed together. In the dark, how many socks must you take out to be sure that you have a matching pair?

Solution. The boxes are the two colors, black and blue. If we take out 3 socks, there are more socks than colors, so two of the socks have the same color. Two socks, on the other hand, might not be enough: we could get one black and one blue sock. So we must take out 3 socks.

Example 3. Prove that among any five integers, there are two whose difference is divisible by 4.

Proof. When an integer is divided by 4, the remainder is 0, 1, 2, or 3. Let the four boxes be these four possible remainders, and place each of the five integers in the box of its remainder. There are more integers than boxes, so two of the integers, call them a and b , have the same remainder r . Then $a = 4p + r$ and $b = 4q + r$ for some integers p and q , and

$$a - b = 4p - 4q = 4(p - q),$$

which is divisible by 4. ■

Example 4. Six different numbers are chosen from the set $\{1, 2, 3, \dots, 10\}$. Prove that two of the chosen numbers add up to 11.

Proof. Here the boxes are not obvious. Split the ten numbers into five pairs, each adding up to 11:

$$\{1, 10\}, \quad \{2, 9\}, \quad \{3, 8\}, \quad \{4, 7\}, \quad \{5, 6\}.$$

Let these five pairs be the boxes, and place each chosen number in the pair that contains it. Six numbers in five boxes: some box contains two of the chosen numbers. Since they are different numbers from the same pair, they are the two numbers of that pair, and their sum is 11. ■

The Extended Pigeonhole Principle

What if there are many more objects than boxes? If 25 people are sorted by birth month, then not just two but at least *three* must share a month: if every month had at most two people, there would be at most 24 people in all. This reasoning works in general.

The Extended Pigeonhole Principle. If more than $k \cdot m$ objects are placed into k boxes, then at least one box contains at least $m + 1$ objects.

The basic principle is the case $m = 1$. The reason is the same as before: if every box contained at most m objects, there would be at most $k \cdot m$ objects in all.

Example 5. Prove that among any 100 people, at least 9 were born in the same month.

Proof. The boxes are the $k = 12$ months. We want some box to contain at least $9 = m + 1$ people, so $m = 8$. Since $12 \cdot 8 = 96$ and we have 100 people, which is more than 96, some month contains at least 9 of them. ■

Example 6. What is the smallest number of people that guarantees that at least 9 of them were born in the same month?

Solution. The boxes are the $k = 12$ months, and $9 = m + 1$, so $m = 8$. Since $12 \cdot 8 = 96$, any group of more than 96 people works. So 97 people are enough. Are 96 people enough? No: it is possible that exactly 8 of them were born in each month, and then no month has 9. So the smallest number is 97.

Example 7. A standard deck has 52 cards in four suits. Cards are drawn from a shuffled deck without looking. How many cards must be drawn to be sure that at least four of them are of the same suit?

Solution. The boxes are the $k = 4$ suits, and we want some box to contain at least $4 = m + 1$ cards, so $m = 3$. Since $4 \cdot 3 = 12$, drawing more than 12 cards guarantees it. So 13 cards are enough. Twelve cards might not be enough: we could get exactly three cards of each suit. The answer is 13.

Example 8. A drawer contains 20 black socks, 16 white socks, 12 yellow socks, and 8 blue socks, all loose (not already in pairs). By a *pair* we mean two socks of the same color. Socks are pulled out of the drawer in the dark. How many socks must be pulled out to be sure that

- (a) there is a pair in our hand,
- (b) there are two pairs in our hand,
- (c) there are two pairs of different colors,
- (d) there are socks of at least two different colors,
- (e) there is a black pair,
- (f) there is a sock of each of the four colors?

Solution. In each part we think about the worst possible luck: the longest run of pulls that still does *not* give us what we want. One more sock then makes it certain.

- (a) The boxes are the four colors. Four socks could all be different colors, but five socks cannot be: two of them must have the same color. The answer is 5.
- (b) Six socks are not enough: three socks of one color and one sock of each of the other colors contain only one pair. With seven socks, there is a pair by part (a). Set that pair aside. The remaining five socks again contain a pair by part (a). The answer is 7.
- (c) Twenty-three socks are not enough: all 20 black socks and one sock of each other color give pairs of only one color. Now take 24 socks and suppose only one color appeared two or more times. That color has at most 20 socks and each other color has at most one, which is at most 23 socks in all, a contradiction. So two different colors each appear at least twice. The answer is 24.
- (d) Black is the color with the most socks, and there are only 20 black socks. So 20 socks might all be black, but 21 socks cannot all have the same color. The answer is 21.
- (e) With bad luck we first pull all $16 + 12 + 8 = 36$ socks that are not black, and then one black sock: 37 socks and still no black pair. The next sock must be black. The answer is 38.
- (f) With bad luck we pull all $20 + 16 + 12 = 48$ black, white, and yellow socks before a single blue one. The next sock must be blue. (Leaving out any other color would use up fewer socks than 48.) The answer is 49.



Practice Problems

1. A class has 40 students. Prove that at least 4 of them were born in the same month.
2. A drawer contains many black, white, and gray socks. How many socks must be taken out, without looking, to be sure of having a matching pair?
3. Prove that among any three integers, there are two whose sum is even. *Hint:* what are the possible boxes?
4. How many cards must be drawn from a standard deck of 52 cards to be sure of getting
 - (a) two cards of the same rank (two aces, or two sevens, and so on)?
 - (b) three cards of the same suit?
5. Prove that among any six integers, there are two whose difference is divisible by 5.

6. Five different numbers are chosen from the set $\{1, 2, 3, \dots, 8\}$. Prove that two of them add up to 9.
7. A bag contains 10 red, 10 blue, and 10 green marbles. How many marbles must be drawn, without looking, to be sure of having at least four of the same color?
8. How many people are needed to be sure that two of them have the same birthday? (A year can have 366 days, counting February 29.) Explain why one fewer person is not enough.
9. A box contains 12 red, 9 green, 6 blue, and 3 yellow pencils. Pencils are taken out of the box in the dark. How many pencils must be taken out to be sure of getting
 - (a) two pencils of the same color,
 - (b) three pencils of the same color,
 - (c) pencils of at least two different colors,
 - (d) two blue pencils,
 - (e) at least one pencil of each of the four colors?



Answers

1. The boxes are the 12 months. Since $12 \cdot 3 = 36$ and $40 > 36$, some month contains at least 4 students.
2. 4 socks. The boxes are the three colors; four socks are more than three colors, so two socks match. Three socks could be one of each color.
3. Use two boxes: even and odd. Three integers in two boxes means two of them are both even or both odd. The sum of two even integers is even, and the sum of two odd integers is even too.
4. a) The boxes are the 13 ranks, so 14 cards. b) The boxes are the 4 suits and $m = 2$; more than $4 \cdot 2 = 8$ cards are needed, so 9 cards.
5. The boxes are the five remainders 0, 1, 2, 3, 4 when dividing by 5. Six integers in five boxes: two have the same remainder r , so they are $5p + r$ and $5q + r$, and their difference $5(p - q)$ is divisible by 5.
6. The boxes are the four pairs $\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$, $\{4, 5\}$, each adding up to 9. Five numbers in four boxes: two of the chosen numbers are in the same pair, so they add up to 9.
7. The boxes are the $k = 3$ colors and $m = 3$. More than $3 \cdot 3 = 9$ marbles are needed, so 10 marbles. Nine marbles could be three of each color.
8. The boxes are the 366 possible birthdays, so 367 people are needed. With 366 people it is possible that all birthdays are different, one for each day of the year.
9. a) 5 b) 9, since more than $4 \cdot 2 = 8$ are needed c) 13, since there are only 12 red pencils d) 26: all $12 + 9 + 3 = 24$ pencils that are not blue, then two blue ones e) 28: all $12 + 9 + 6 = 27$ red, green, and blue pencils, then a yellow one