

Probability is the mathematics of uncertainty. Some things we cannot predict with certainty – the roll of a die, the toss of a coin, the card on top of a shuffled deck – but we can still say how *likely* each possibility is, and we do it with a number. In this handout we set up the basic vocabulary.

Definition. An **experiment** is any process whose result cannot be predicted with certainty in advance. A possible result of the experiment is called an **outcome**. The set of all possible outcomes is called the **sample space**, and we denote it by S .

Example 1. Describe the sample space for each experiment.

- (a) Toss a coin.
- (b) Roll a standard six-sided die.
- (c) Toss two coins, one after the other.



Solution.

- (a) $S = \{H, T\}$, where H stands for heads and T for tails.
- (b) $S = \{1, 2, 3, 4, 5, 6\}$.
- (c) We record the result of the first coin, then the second: $S = \{HH, HT, TH, TT\}$. Notice that HT and TH are two *different* outcomes: they are not the same, because in one the first coin showed heads and in the other it showed tails.

Usually we are interested in more than a single outcome. We may want to know, for instance, whether the die shows an *even* number, and that can happen in several different ways.

Definition. An **event** is a collection of outcomes, that is, a subset of the sample space S . We say that an event **occurs** if the outcome of the experiment is one of the outcomes in the event.

Example 2. A die is rolled. List the outcomes in each event.

- (a) E : the number rolled is even.
- (b) F : the number rolled is greater than 4.
- (c) G : the number rolled is 7.



Solution.

- (a) $E = \{2, 4, 6\}$.
- (b) $F = \{5, 6\}$.
- (c) $G = \{ \}$, the empty set: no outcome of the experiment is a 7.

Two events deserve special names. An event that contains *no* outcomes, like G above, is called an **impossible event**: it can never occur. The event S itself, which contains every outcome, is called a **certain event**: it always occurs, no matter which outcome the experiment produces. (For example, when a die is rolled, the event “the number rolled is between 1 and 6” is certain.)

The Probability of an Event

How should we measure how likely an event is? In many experiments there is a good reason to believe that every outcome is just as likely as any other. A fair coin is as likely to land heads as tails, and a fair die is as likely to land on a 1 as on a 6. We call such outcomes **equally likely**. When all outcomes in the sample space are equally likely, we measure the likelihood of an event by the *fraction* of the outcomes that belong to it.

Definition. Suppose an experiment has a finite sample space S in which all outcomes are equally likely. The **probability** of an event E is

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{n(E)}{n(S)}.$$

Example 3. A fair die is rolled. Find the probability of each event from the previous example.

Solution. The sample space has $n(S) = 6$ equally likely outcomes.

$$P(E) = \frac{3}{6} = \frac{1}{2}, \quad P(F) = \frac{2}{6} = \frac{1}{3}, \quad P(G) = \frac{0}{6} = 0.$$

This definition tells us what values a probability can take. The number of outcomes in an event can be as small as 0 and can be at most the number of outcomes in S , so the fraction is never negative and never more than 1:

$$0 \leq P(E) \leq 1.$$

An impossible event has probability 0, and a certain event has probability 1. A probability near 0 means the event is unlikely, and a probability near 1 means it is very likely.

Example 4. A fair die is tossed twice. Find the probability that the sum of the two numbers is 7.

Solution. An outcome is an ordered pair (first toss, second toss), such as $(2, 5)$, which means a 2 on the first toss and a 5 on the second. There are 6 possibilities for the first toss and 6 for the second, so there are $n(S) = 6 \cdot 6 = 36$ equally likely outcomes. Here they all are:

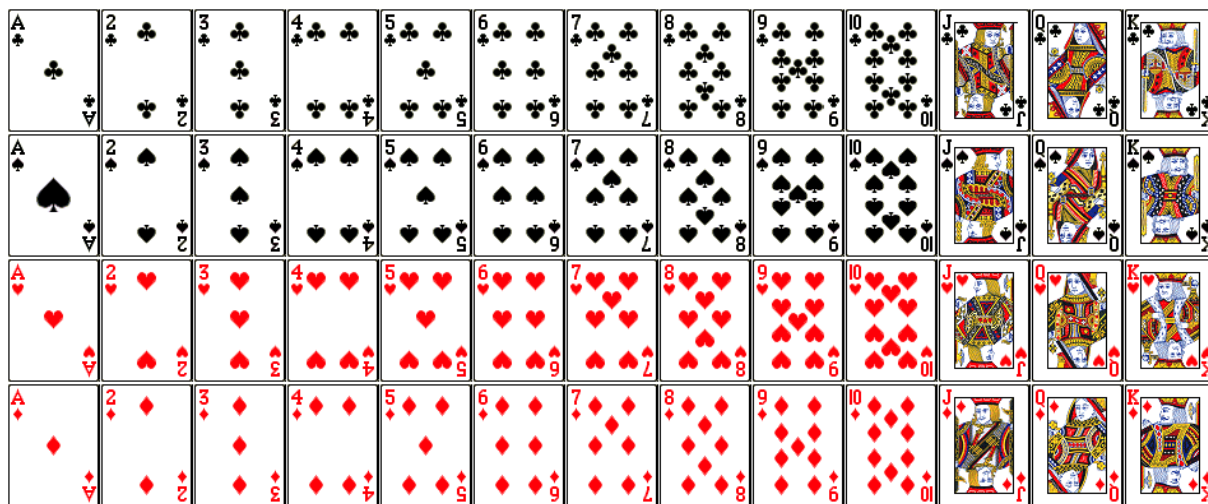
(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The outcomes whose sum is 7 are shown in bold: (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1). There are 6 of them, so

$$P(\text{sum is } 7) = \frac{6}{36} = \frac{1}{6}.$$

Be careful. The definition applies only when the outcomes are *equally likely*. When two coins are tossed, the number of heads can be 0, 1, or 2, but these three results are *not* equally likely: getting exactly one head can happen in two ways (*HT* or *TH*), while getting no heads can happen in only one way (*TT*). To use the definition, we must list the four equally likely outcomes *HH*, *HT*, *TH*, *TT*.

A standard deck has 52 cards: four suits (clubs, spades, hearts, diamonds), each with thirteen ranks. The jacks, queens, and kings are the 12 face cards.



Practice Problems

- List the sample space for each experiment.
 - A letter is chosen at random from the word MATH.
 - A coin is tossed three times.
- A fair die is rolled. Find the probability of each event.
 - The number rolled is a prime number.
 - The number rolled is a multiple of 3.
 - The number rolled is at most 4.
- A bag contains 5 red, 3 blue, and 2 green marbles. One marble is drawn at random. Find the probability that it is
 - blue,
 - red or green,
 - yellow.

4. A fair die is tossed twice. Find the probability that
 - (a) the sum is 8,
 - (b) both tosses show the same number,
 - (c) the sum is less than 13.
5. A fair coin is tossed three times. Find the probability of getting exactly two heads.
6. One card is drawn at random from a standard deck of 52 cards. Find the probability that it is
 - (a) a heart,
 - (b) a jack, queen, or king,
 - (c) the ace of spades or the ace of hearts.
7. A student says: “When I toss two coins, I get 0, 1, or 2 heads, so each of these has probability $\frac{1}{3}$.” Is this correct? Explain, and find the actual probabilities.
8. Two students missed a chemistry exam, claiming that they had a flat tire. The professor gives them a makeup exam in separate rooms. The first question, worth 5 points, is an easy chemistry question. The second question, worth 95 points, is: “Which tire?” Suppose that each student picks one of the car’s four tires at random. What is the probability that their answers match?



Answers

1. a) $S = \{M, A, T, H\}$ b) $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ (8 outcomes)
2. a) $\{2, 3, 5\}$, so $\frac{3}{6} = \frac{1}{2}$ b) $\{3, 6\}$, so $\frac{2}{6} = \frac{1}{3}$ c) $\{1, 2, 3, 4\}$, so $\frac{4}{6} = \frac{2}{3}$
3. There are 10 marbles. a) $\frac{3}{10}$ b) $\frac{5+2}{10} = \frac{7}{10}$ c) $\frac{0}{10} = 0$
4. There are 36 outcomes. a) The sum is 8 for $(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)$, so $\frac{5}{36}$. b) $(1, 1), (2, 2), \dots, (6, 6)$, so $\frac{6}{36} = \frac{1}{6}$. c) Every sum is at most 12, so this event is certain: $\frac{36}{36} = 1$.
5. The outcomes with exactly two heads are HHT, HTH, THH , so the probability is $\frac{3}{8}$.
6. a) $\frac{13}{52} = \frac{1}{4}$ b) $\frac{12}{52} = \frac{3}{13}$ c) $\frac{2}{52} = \frac{1}{26}$
7. No. The three results are not equally likely. The equally likely outcomes are HH, HT, TH, TT , so $P(0 \text{ heads}) = \frac{1}{4}$, $P(1 \text{ head}) = \frac{2}{4} = \frac{1}{2}$, and $P(2 \text{ heads}) = \frac{1}{4}$.
8. Call the tires LF, RF, LR, and RR (left or right, front or rear). An outcome is a pair (first student’s answer, second student’s answer), and there are $4 \cdot 4 = 16$ equally likely outcomes. The answers match in 4 of them: $(LF, LF), (RF, RF), (LR, LR), (RR, RR)$. So the probability is $\frac{4}{16} = \frac{1}{4}$.